Grading on $D_{R|K}$

The polynomial ring R has a standard grading via $|x_i| = 1$ for each i. This induces a grading on $D_{R|K}$ via $|\delta| = k$ if $\delta([R]_a) \subseteq [R]_{a+k}$ for each a. (homogeneous of deg a)

E.g., $\left|\frac{\partial}{\partial x_i}\right| = -1$, $|\bar{x}_i| = 1$.

Of course, as with graded rings in general, not every operator is homogeneous, but every op. is a sum of homog. operators in a unique way.

II. Application: Invariants of Classical Groups

Perhaps the first application historically of differential operators in Commutative Algebra is to computation of invariant rings. This goes back to 19th century (Gordan, Cayley, Hilbert, etc.), and was recapped nicely in Weyl's Classical Groups.

We don't need to know much about differential operators; this is a bare bones application of the philosophy that differential operators undo multiplication / decrease order in a structured way.

We will let K be a field of characteristic zero, and

$$R = K\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$$

be a polynomial ring in $2n$ variables.

For short, we'll write $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$.

Some particular differential operators

- The Euler operator on our polynomial ring is
  $$E = \sum_i \bar{x}_i \frac{\partial}{\partial x_i} + \sum_i \bar{y}_i \frac{\partial}{\partial y_i},$$
  and likewise on any polynomial ring. The point is Euler's identity: if f is homogeneous of degree d, we have
  $$E(f) = d f.$$
  To prove it, we check for monomials:
  $$E(x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_n^{\beta_n}) = \sum_i \alpha_i \underline{x}^{\underline{\alpha}} \underline{y}^{\underline{\beta}} + \sum_i \beta_i \underline{x}^{\underline{\alpha}} \underline{y}^{\underline{\beta}} = \left(\sum_i \alpha_i + \sum_i \beta_i\right)(\underline{x}^{\underline{\alpha}} \underline{y}^{\underline{\beta}}).$$

In our situation, we have finer gradings that relate to other operators. We give R an $\mathbb{N}^n$-grading by setting $|\bar{x}_i| = |\bar{y}_i| = (0, \dots, 0, \underset{i^{\text{th}}\text{ place}}{1}, 0, \dots, 0) =: e_i$

for each i.

- Set $E_{ii} = \bar{x}_i \frac{\partial}{\partial x_i} + \bar{y}_i \frac{\partial}{\partial y_i}$ for each i.
  If f is homogeneous in the $\mathbb{N}^n$-grading, with degree $(d_1, \dots, d_n)$, then
  $$E_{ii}(f) = d_i \cdot f.$$
  Check in the same way as above.

- Set $E_{ij} = \bar{x}_i \frac{\partial}{\partial x_j} + \bar{y}_i \frac{\partial}{\partial y_j}$ for $i \neq j$.
  This is called a polarization operator.
  Intuitively, this turns j variables into i variables; more precisely, if $|f| = (d_1, \dots, d_i, \dots, d_j, \dots, d_n)$ then $|E_{ij}(f)| = (d_1, \dots, d_i + 1, \dots, d_j - 1, \dots, d_n)$.

- Set $D_{ij} = \det\begin{pmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{pmatrix} = \frac{\partial}{\partial x_i} \circ \frac{\partial}{\partial y_j} - \frac{\partial}{\partial y_i} \circ \frac{\partial}{\partial x_j}$ for $i \neq j$.

We will need the observation that E_{ii} and E_{ij} are <u>derivations</u>: k-linear maps δ that satisfy $\delta(fg) = f\delta(g) + g\delta(f)$ for $f, g \in R$. Indeed, $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ all satisfy this product rule, and any $\bar{R}$-linear combination of derivations is a derivation.

<u>Capelli's identity</u> [switch E_{jj} & E_{ii}...]

<u>Thm</u> (Capelli): $\det\left(\begin{bmatrix} E_{jj}+1 & E_{ij} \\ E_{ji} & E_{ii} \end{bmatrix}\right) = \det\left(\begin{bmatrix} \bar{x}_i & \bar{x}_j \\ \bar{y}_i & \bar{y}_j \end{bmatrix}\right) \circ \det\left(\begin{bmatrix} \partial/\partial x_i & \partial/\partial x_j \\ \partial/\partial y_i & \partial/\partial y_j \end{bmatrix}\right)$.

<u>pf</u> We compute using the relations on the Weyl algebra.

$$(E_{jj}+1)\circ E_{ii} - E_{ji}\circ E_{ij} = \left(\bar{x}_j \frac{\partial}{\partial x_j} + \bar{y}_j \frac{\partial}{\partial y_j} + 1\right)\cdot\left(\bar{x}_i \frac{\partial}{\partial x_i} + \bar{y}_i \frac{\partial}{\partial y_i}\right) - \left(\bar{x}_j \frac{\partial}{\partial x_i} + \bar{y}_j \frac{\partial}{\partial y_i}\right)\left(\bar{x}_i \frac{\partial}{\partial x_j} + \bar{y}_i \frac{\partial}{\partial y_j}\right)$$

$$= \bar{x}_j \frac{\partial}{\partial x_j} \bar{x}_i \frac{\partial}{\partial x_i} + \bar{x}_j \frac{\partial}{\partial x_j} \bar{y}_i \frac{\partial}{\partial y_i} + \bar{y}_j \frac{\partial}{\partial y_j} \bar{x}_i \frac{\partial}{\partial x_i} + \bar{y}_j \frac{\partial}{\partial y_j} \bar{y}_i \frac{\partial}{\partial y_i} + \bar{x}_i \frac{\partial}{\partial x_i} + \bar{y}_i \frac{\partial}{\partial y_i}$$
$$- \bar{x}_j \frac{\partial}{\partial x_i} \bar{x}_i \frac{\partial}{\partial x_j} - \bar{x}_j \frac{\partial}{\partial x_i} \bar{y}_i \frac{\partial}{\partial y_j} - \bar{y}_j \frac{\partial}{\partial y_i} \bar{x}_i \frac{\partial}{\partial x_j} - \bar{y}_j \frac{\partial}{\partial y_i} \bar{y}_i \frac{\partial}{\partial y_j}$$

$$= \cancel{\bar{x}_i \bar{x}_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}} + \bar{x}_j \bar{y}_i \frac{\partial}{\partial x_j} \frac{\partial}{\partial y_i} + \bar{x}_i \bar{y}_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} + \cancel{\bar{y}_i \bar{y}_j \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j}} + \cancel{\bar{x}_i \frac{\partial}{\partial x_i}} + \cancel{\bar{y}_i \frac{\partial}{\partial y_j}}$$
$$- \cancel{\bar{x}_i \bar{x}_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}} - \bar{x}_j \frac{\partial}{\partial x_j} - x_j \bar{y}_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} - x_i y_j \frac{\partial}{\partial x_j} \frac{\partial}{\partial y_i} - \cancel{\bar{y}_i \bar{y}_j \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j}} - \bar{y}_j \frac{\partial}{\partial y_j}$$

(modulo switch i, j ...)

$$= (\bar{x}_i \bar{y}_j - \bar{x}_j \bar{y}_i)\left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} - \frac{\partial}{\partial x_j} \frac{\partial}{\partial y_i}\right) \quad \blacksquare$$

$$\det\begin{pmatrix} E_{ii}+I & E_{ij} \\ E_{ji} & E_{jj} \end{pmatrix}$$

$$\parallel$$

$$(E_{ii}+1)\circ E_{jj} - E_{ji}\circ E_{ij} = \Big(\bar{x}_i \frac{\partial}{\partial x_i} + \underbrace{\bar{y}_i \frac{\partial}{\partial y_i} + 1\Big)\circ\Big(\bar{x}_j \frac{\partial}{\partial x_j}}_{\text{all terms commute in F\&L ing}} + \bar{y}_j \frac{\partial}{\partial y_j}\Big)$$

$$- \Big(\underbrace{\bar{x}_j \frac{\partial}{\partial x_i} + \bar{y}_j \frac{\partial}{\partial y_i}\Big)\circ\Big(\bar{x}_i \frac{\partial}{\partial x_j} + \bar{y}_i \frac{\partial}{\partial y_j}}_{\text{noncommuting terms from F\&L.}}\Big)$$

e.g. $\bar{x}_j \frac{\partial}{\partial x_i} \bar{x}_i \frac{\partial}{\partial x_j}$

$= \bar{x}_i \bar{x}_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \cancel{\bar{x}_j} \frac{\partial}{\partial x_j}$

$\cancel{}$ ...

$$= \underline{\bar{x}_j \bar{y}_i \left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial y_i} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}\right) + \bar{x}_i \bar{y}_j \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial y_i}\right)}$$

$= \cancel{}$ RHS.

SL_2-action

Let $G = SL_2(k)$ act on R by

$$g \cdot \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} = g \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \quad \text{matrix product}$$

i.e., $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow g \cdot \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} = \begin{bmatrix} a\underline{x} + b\underline{y} \\ c\underline{x} + d\underline{y} \end{bmatrix}$,

so $x_i \mapsto ax_i + by_i$
$y_i \mapsto cx_i + dy_i$.

We want to compute all the invariants of this action.

Set $\bar{\Delta}_{ij} = \det \begin{bmatrix} \bar{x}_i & \bar{x}_j \\ \bar{y}_i & \bar{y}_j \end{bmatrix} = \bar{x}_i \bar{y}_j - \bar{x}_j \bar{y}_i$.
(and $\Delta_{ij} = x_i y_j - x_j y_i$)

Each Δ_{ij} is invariant, since multiplying by a det-1 matrix doesn't ~~affect~~ the det. of this submatrix.

Thus, $S := k[\{\Delta_{ij} \mid i<j\}] \subseteq R^G$ is a subring of the invariant ring.

We will show that actually $S = R^G$.

Lem: S and R^G are $\mathbb{N}^n$-graded subrings of R.

pf: For S, it is clear since its generators are homogeneous. For R^G, we note that the action of G preserves degrees in the $\mathbb{N}^n$-grading, so the lemma follows.

Lem: We have (i) $E_{ij}(R^G) \subseteq R^G \qquad i \leq j$

(ii) $E_{ij}(S) \subseteq S \qquad i \leq j$

(iii) $D_{ij}(R^G) \subseteq R^G \qquad i < j.$

pf: ~~In each case, it suffices to show that the claim holds for homogeneous elements in the grading, by the last lemma.~~

(i) First, we show that $E_{ij} \circ g = g \circ E_{ij}$ for $g \in SL_2(k)$.

Write $\underline{x}' = g(\underline{x}) = a\underline{x} + b\underline{y}$ and $\underline{y}' = g(\underline{y}) = c\underline{x} + d\underline{y}$. We have

$$(g \circ E_{ij})(f(\underline{x},\underline{y})) = g\left(x_i \frac{\partial f}{\partial x_j}(\underline{x},\underline{y}) + y_i \frac{\partial f}{\partial y_j}(\underline{x},\underline{y})\right)$$
$$= x_i' \frac{\partial f}{\partial x_j}(\underline{x}',\underline{y}') + y_i' \frac{\partial f}{\partial y_j}(\underline{x}',\underline{y}'), \text{ and}$$

$$(E_{ij} \circ g)(f(\underline{x},\underline{y})) = x_i \frac{\partial}{\partial x_j}(f(\underline{x}',\underline{y}')) + y_i \frac{\partial}{\partial y_j}(f(\underline{x}',\underline{y}'))$$
$$= (ax_i + by_i)\frac{\partial f}{\partial x_j}(\underline{x}',\underline{y}') + (cx_i + dy_i)\frac{\partial f}{\partial y_j}(\underline{x}',\underline{y}') \quad \text{(chain rule)}$$
$$= x_i' \frac{\partial f}{\partial x_j}(\underline{x}',\underline{y}') + y_i' \frac{\partial f}{\partial y_j}(\underline{x}',\underline{y}').$$

(iii) Similar.

(ii) We use that E_{ij}'s are derivations.

First, ~~$E_{ij}(\Delta_{kl}) = \ldots$~~

$\{i,j\} \cap \{k,l\} = \emptyset \qquad \{i,j\} \cap \{k,l\} = \{j\} = l \qquad \{i,j\} = \{k,l\}$

compute $E_{ij}(\Delta_{kl}) \in \{0, \Delta_{ik}, \pm 2\}$

for various cases so

$E_{ij}(\Delta_{kl}) \in S.$

Now, use the product rule to conclude that E_{ij} of any product of these is again in S.

Lem: There are no (nonconstant) homogeneous invariants of degree $(d, 0, \ldots, 0)$.

Th'm: $R^{SL_2(k)} = k[\{\Delta_{ij}\}]$; i.e., $S = R^G$.

Pf: It suffices to show that any homogeneous element of R^G is in S. If not, pick $f \in R^G$ homog$^{(d_1,\dots,d_n)}$ of minimal total degree $= d = d_1 + \dots + d_n$, and among these, with d_1 maximal.

We know $d_1 \neq d$ by previous Lemma, so $d_j \neq 0$ for some j. Then,

$$((E_{11}+1)E_{jj})(f) = E_{j1} \circ E_{1j}(f) + \Delta_{1j} \circ D_{1j}(f).$$

We have $D_{1j}(f) \in R^G$, and f of min deg not in $S \Rightarrow f \in S$, and $\Delta_{ij} \in S$.

$((E_{11}+1) \circ E_{jj})(f) = (d_1+1)(d_j) f$ is a nonzero scalar multiple of f.

The first component of degree of $E_{1j}(f)$ is greater than that of f, and total degree is same, so $E_{1j}(f) \in S$ (because in R^G), and $(E_{j1} \circ E_{1j})(f) \in S$ by Lemma. Thus, LHS $\in S$, so $f \in S$. $\square$