1

Differential Operators

Mon, Wed 9:30–10:50

no class 27/1, 29/1, 3/2

yes class 22/1, 5/2, etc.

0. Introduction

This class is on differential operators (both smooth & singular settings) and various applications to CA.

Basic example of a differential operator: on a polynomial ring $K[x_1, \dots, x_n]$ over a field K, $\delta = \frac{\partial}{\partial x_1}$ is a differential operator

– K doesn't need to be $\mathbb{R}$ or $\mathbb{C}$; use linearity & product rule /power rule instead of analysis.

In general, given an A-algebra R (A, R commut.) we will have a filtered noncommutative ring of differential operators:

$$A \to R \quad \rightsquigarrow \quad D_{R|A}$$

$$D^0_{R|A} \subseteq D^1_{R|A} \subseteq D^2_{R|A} \subseteq \cdots$$

The elements operate on R:

$$D_{R|A} \circlearrowright R.$$

R is a $D_{R|A}$-module.

This action is a way to undo multiplication. (or at least to try to!) in a way that preserves structure

eg.,

in $R=\mathbb{Q}[x]$, multiplication by $x^n : 1 \mapsto x^n$

and no R-operation can send $x^n \mapsto 1$, but

in $D_{R|K}$, have $(\frac{\partial}{\partial x})^n$

$$1 \xmapsto{\cdot x^n} x^n \xmapsto{(\frac{\partial}{\partial x})^n} n! \xmapsto{\frac{1}{n!}} 1$$

and $(\frac{\partial}{\partial x})^n$ isn't a random function sending $x^n \mapsto n!$; it's a function we understand well.

More algebraically, $1 \notin R(x^n)R$, but $1 \in D_{R|K}\cdot(x^n)$.

Along similar lines, R is not a simple R-module, but R is a simple $D_{R|K}$-module

$\Rightarrow$ large R-modules often become small $D_{R|K}$-modules.

Will develop general theory of diff. operators for algebras $A \to R$. In general, it is hard to compute the rings $D_{R|A}$ and to understand what good properties they have or don't have.

At least when $R = A[x_1, \ldots, x_n]$ poly ring, can compute & establish many good properties.

If also we assume A is a field of char 0, can say extremely strong & specialized properties of $D_{R|A}$.

There is a lot of research on this that touches many fields of math, and many extremely deep results & complicated machinery. Will develop just a bit of this at end of class, but focus more on more general situation.

What applications?

Symbolic powers: Given a prime ideal $P \subseteq R$, we have $P^{(n)} := P^n \cdot R_P \cap R$.

Will give a very concrete characterization ~~of~~ $P^{(n)}$ in poly ring R, and a characterization of $P^{(n)}$ more generally, in terms of diff. operators.

Singularities: A local ring $(R, \mathfrak{m}, k)$ satisfies the inequality $\dim R \le \dim_k(\mathfrak{m}/\mathfrak{m}^2)$

(= min'l number of gens of $\mathfrak{m}$)

R is regular if "=" holds, and singular otherwise.

Will use ~~diff~~ properties of $D_{R|A}$ to characterize where R is regular, and various classes of singularities.

Local cohomology: Given $(f_1, \ldots, f_n) = I$ ideal in R, we define the local cohomology modules of R with support in I as

$$H^i_I(R) = H^i\left(0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{i<j} R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_n} \to 0\right)$$

with maps as ± 1's in each component in such a way as to make a complex.

The modules R_{f_i} and $H^i_I(R)$ are usually not f.g. R-modules. When R is a poly ring over field of char 0, will find that R_{f_i}, $H^i_I(R)$ have finite length as $D_{R|K}$-modules, and finitely many asso primes as R-modules.

And more...

I. Differential operators on poly rings in char 0

Soon we will give a definition of diff. ops in general, but we first give a special def. in the most special case.

Def: Let K be a field of characteristic zero, and $R = K[x_1, \dots, x_n]$ be a polynomial ring. The ring of K-linear differential operators on R is the subring of $\mathrm{End}_K(R)$ (K-linear endomorphisms of R with composition as multiplication) generated by

K, $\overline{x_1}, \dots, \overline{x_n}$ (the endomorphisms $\overline{x_i}: f \mapsto x_i \cdot f$), and $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$; i.e.,

where $\frac{\partial}{\partial x_i}$ is the K-linear map given by $\frac{\partial}{\partial x_i}(x_1^{a_1} \cdots x_i^{a_i} \cdots x_n^{a_n}) = a_i\, x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_n^{a_n}$

$$D_{R|K} = K\langle \overline{x_1}, \dots, \overline{x_n}, \tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_n} \rangle.$$

Observations:

1) Since $D_{R|K} \subseteq \mathrm{End}_K(R)$, every element of $D_{R|K}$ is a K-v.s. endomorphism of R. This makes R into a left $D_{R|K}$-module in a canonical way, where the module action is "apply the endomorphism." Let's unpackage this once just so we're 100% sure that we want "left" instead of "right."

$\alpha, \beta \in D_{R|K} (\subseteq \mathrm{End}_K(R))$, $r \in R$

$$(\underset{D_{R|K}\text{-mult.}}{\alpha \cdot \beta}) \underset{\substack{D_{R|K} \curvearrowright R \\ \text{action}}}{\cdot} r := (\alpha \circ \beta)(r) = \alpha(\beta(r))$$

$$= \alpha(\beta \underset{\substack{D_{R|K} \curvearrowright R \\ \text{action}}}{\cdot} r) = \alpha \underset{\substack{D_{R|K} \curvearrowright R \\ \text{action}}}{\cdot} (\beta \cdot r),$$

and $(\alpha + \beta) \cdot r := (\alpha + \beta)(r) = \alpha(r) + \beta(r)$

$$= (\alpha \cdot r) + (\beta \cdot r).$$

We will use both module action notation and function notation.

2) We have $K\langle \overline{x_1}, \dots, \overline{x_n} \rangle \subseteq D_{R|K}$ by definition. Each map $\overline{x_i}$ is in fact R-linear, so $K\langle \overline{x_1}, \dots, \overline{x_n} \rangle \subseteq \mathrm{End}_R(R)$, which is $= \mathrm{Hom}_R(R, R) \underset{R\text{-modules.}}{\cong} R$

In fact $K\langle \overline{x_1}, \dots, \overline{x_n} \rangle \cong K[x_1, \dots, x_n] = R$ as rings by the obvious map. $(\overline{x_i} \leftarrow\!\shortmid x_i)$.

Thus, there is an injective K-algebra homomorphism

$$R \hookrightarrow D_{R|K}$$

sending r to "multiplication by r."

We should excercise some caution in distinguishing elements in R from their images in $D_{R|K}$ (the multiplications).

For example, $\underset{D_{R|K}}{\left(\frac{\partial}{\partial x}\right)^2} \underset{\substack{D \curvearrowright R \\ \text{action}}}{\cdot} \underset{R}{x} = 0$

~~But~~ $\left(\frac{\partial}{\partial x}\right)^2 \cdot \overline{x} \neq 0$

$D_{R|K}$ $\underset{\text{mult.}}{D_{R|K}}$ $D_{R|K}$

since $\left(\frac{\partial}{\partial x}\right)^2 \overline{x}\,(x) = \left(\frac{\partial}{\partial x}\right)^2 (x^2) = 2$,

so $\left(\frac{\partial}{\partial x}\right)^2 \overline{x}$ is not the 0 endomorphism of R.

We won't worry so much for elements of K, since they commute with everything

Relations & standard forms

To try to understand $D_{R|K}$, we want to find relations between its generators. Some generators commute:

$$\begin{cases} \overline{x_i}\,\overline{x_j} = \overline{x_j}\,\overline{x_i} & i \neq j \\ \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} & i \neq j \\ \overline{x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\overline{x_i} & i \neq j \end{cases}$$

Indeed, it suffices to check these on a K-v.s. basis for R, e.g. the monomials, where it is clear.

We now ~~want~~ need to see how variables and partial derivatives commute or fail to; we have

$$\frac{\partial}{\partial x_i}\overline{x_i}\,(x_1^{a_1}\cdots x_i^{a_i}\cdots x_n^{a_n}) = \frac{\partial}{\partial x_i}(x_1^{a_1}\cdots x_i^{a_i+1}\cdots x_n^{a_n})$$
$$= (a_i+1)(x_1^{a_1}\cdots x_i^{a_i}\cdots x_n^{a_n}),$$

so $\frac{\partial}{\partial x_i}\overline{x_i} = \overline{a_i+1}$, whereas

$$\overline{x_i}\frac{\partial}{\partial x_i}(x_1^{a_1}\cdots x_i^{a_i}\cdots x_n^{a_n}) = \overline{x_i}\{a_i x_1^{a_1}\cdots x_i^{a_i-1}\cdots x_n^{a_n}\}$$
$$= a_i x_1^{a_1}\cdots x_i^{a_i}\cdots x_n^{a_n},$$

so $\overline{x_i}\frac{\partial}{\partial x_i} = \overline{a_i}$.

In particular, $\frac{\partial}{\partial x_i}\overline{x_i} = \overline{x_i}\frac{\partial}{\partial x_i} + \overline{1}$ for each i.