Some old qualifying exam questions

Here are some old qualifying exam problems you are already ready to solve.

Problem 1. Let H be a subgroup of a group G. Show that $[G, G] \leq H$ if and only if H is a normal subgroup of G and G/H is abelian.

Problem 2. Let G be a group and H a subgroup of G. The centralizer of H in G is the set of elements of G that commute with each element of H:

$$C_G(H) =: \{g \in G \mid gh = hg \text{ for all } h \in H \ \}.$$

Prove that if H is normal in G, then $C_G(H)$ is a normal subgroup of G and that $G/C_G(H)$ is isomorphic to a subgroup of the automorphism group of H.

Problem 3. Let G be a group (not necessarily finite) and H a nonempty subset of G that is closed under multiplication. Suppose that for all $g \in G$ we have $g^2 \in H$.

(a) Show that H must be a subgroup of G.

(b) Show that H must be a normal subgroup of G.

(c) Show that G/H must be abelian.

Problem 4. Fix a prime number p, and let A denote the abelian group of all complex roots of unity whose orders are powers of p; that is,

$$A = \{z \in \mathbb{C} \mid z^{p^n} = 1 \text{ for some integer } n \geq 1\}.$$

Prove the following statements.

(a) Every non-trivial subgroup of A contains the group of p-th roots of unity.

(b) Every proper subgroup of A is cyclic.

(c) If B and C are subgroups of A, then either $B \subseteq C$ or $C \subseteq B$.

(d) For each $n \geq 0$, there exists a unique subgroup of A with p^n elements.

Problem 5. Let G be a group and H and K subgroups. Recall that HK is the subset of G defined as $HK = \{hk \mid h \in H, k \in K\}$.

(a) Prove $HK = KH$ if and only if HK is a subgroup of G.

(b) Give an example (with justification) where HK is not a subgroup.

Problem 6. Let G be a finite group.

(a) If N is a normal subgroup of G and $\#N = 2$, prove that N is contained in the center $Z(G)$ of G.

(b) Suppose that $\#Z(G)$ is odd and that G contains a non-trivial simple[1] subgroup H with $[G : H] = 2$. Prove that H is the only non-trivial proper normal subgroup of G.

[1] H is **simple** provided it has no normal subgroups other than $\{e\}$ and H itself.