Some old qualifying exam questions on rings

Here are some old qualifying exam problems you will be ready to solve.

Problem 1. Let R be a commutative ring and x an indeterminant. Prove that $R[x]$ is a principal ideal domain (PID) if and only if R is a field.

Problem 2. Let d be a square free integer and $\mathbb{Q}(\sqrt{d})$ the subring of $\mathbb{C}$ defined by $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$. Show that there is a ring isomorphism $\mathbb{Q}[x]/(x^2 - d) \cong \mathbb{Q}(\sqrt{d})$.

Problem 3. (a) Prove that a finite integral domain must be a field.

(b) Prove that if R is a commutative ring and $P \subseteq R$ is a prime ideal such that P has finite index as a subgroup of $(R, +)$, then P is a maximal ideal. Give an example to show that this implication may fail if the finite index assumption is dropped.

Problem 4. Let R be a commutative ring, and set $I = \{r \in R \mid r^n = 0 \text{ for some integer } n \geq 1\}$. Prove that following assertions.

(a) I is an ideal in R.

(b) If R/I is a field, then each element of R is either a unit or in I.

Problem 5. Let I be a nonzero ideal of the ring of Gaussian integers $\mathbb{Z}[i]$. Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite.

Problem 6. On UFDs.

(a) Prove that in a UFD a nonzero element p is irreducible if and only if the ideal it generates (p) is a prime ideal.

(b) Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.