Some old qualifying exam questions on groups

Here are some old qualifying exam problems you are already ready to solve.

Problem 1. (a) Let G be a simple group of order 60. Determine the number of elements of G of order 5.

(b) Show that there is no simple group of order 30.

Problem 2.

A non-abelian group of order 27.

(a) Prove there exists a non-abelian group of 27. (*Hint:* Use a semi-direct product.)

(b) Find, with justification, a presentation of the group you found in part (a).

Problem 3. Determine, with justification, all the isomorphism classes of groups of order 45.

Problem 4. Let $G = A_7$ and S be the set of all elements of G of order 7. Prove that S is not a conjugacy class of G.

Problem 5. Suppose H is a subgroup of a group G and $[G : H] = 7$.

(a) Prove G contains a normal subgroup N such that $N \subseteq H$ and $[G : N] \leq 7!$.

(b) Prove 7! is the best possible bound for the previous part — i.e., prove there is a group G and a subgroup H with $[G : H] = 7$ such that for every normal subgroup N of G with $N \subseteq H$, we have $[G : N] \geq 7!$.

Problem 6. Consider the following statement, which can be viewed as a converse to Lagrange's theorem:

> Let G be a finite group of order n and d a positive divisor of n. Then G has a subgroup of order d.

(a) Give a counterexample (with complete justification) to the above statement.

(b) Prove that if one adds the hypothesis that G is abelian, then the above statement is true.

Problem 7. Let G be a group and K a finite cyclic normal subgroup of G. Prove that $G' \subseteq C_G(K)$, where G' is the commutator subgroup of G and $C_G(K) = \{g \in G \mid gk = kg \text{ for all } k \in K\}$. (Hint: Consider an appropriate action of G on K.)

Problem 8. Prove that no group of order 150 is simple.