DEFINITION: Let R be a ring.

(i) An ideal I of R is a **maximal ideal** if I is proper and for any proper ideal J, $I \subseteq J$ implies $I = J$. That is, I is maximal under containment among all proper ideals of R.

(ii) Let R be commutative. An ideal I of R is a **prime ideal** if I is proper and $ab \in I$ implies $a \in I$ or $b \in I$.

THEOREM 1: Let R be a commutative ring and I an ideal.

(i) The ideal I is maximal if and only if R/I is a field.

(ii) The ideal I is prime if and only if R/I is an integral domain.

(1) Prime ideals vs maximal ideals:

(a) Use Theorem 1 to quickly explain why every maximal ideal in a commutative ring is prime.

We have I is maximal implies R/I is a field, which implies R/I is a domain, which implies I is prime.

(b) Show that the ideal (2) in $\mathbb{Z}[x]$ is prime but not maximal.

$\mathbb{Z}[x]/(2) \cong \mathbb{Z}/2[x]$, which is a domain but not a field.

(c) Identify a maximal ideal in $\mathbb{Z}[x]$.

The ideal $(2, x)$ is maximal, since $\mathbb{Z}[x]/(2, x) \cong \mathbb{Z}/2$.

(2) Prove[1] Theorem 1.

(i) By the Lattice Isomorphism Theorem, the ideals of R/Q are of the form I/Q, where I is an ideal in R containing Q.

By an exercise, R/Q is a field if and only if R/Q has only two ideals, $\{0\} = Q/Q$ and R/Q. Thus R/Q is a field if and only if the only ideals that contain Q are Q and R.

(ii) Now suppose Q is prime. If

$$(r + I)(r' + I) = 0 + I,$$

then $rr' \in I$ and hence either $r \in I$ or $r' \in I$, so that either

$$r + I = 0 \quad \text{or} \quad r' + I = 0.$$

Since R is commutative, then R/I is also commutative, and since Q is a proper, then R/I is not the zero ring. This proves that R/Q is a domain.

Conversely, suppose that R/Q is a domain. Since R/Q is not the zero ring, Q is proper. If $x, y \in R$ satisfy $xy \in I$, then

$$(x + I)(y + I) = 0$$

in R/Q, and hence either $x + Q = 0$ or $y + Q = 0$. It follows $x \in Q$ or $y \in Q$. This proves that Q is prime.

[1] Hint: For part (i), you might want use a HW problem characterizing fields in terms of ideals.

THEOREM 2: Let R be a ring. Then R has a maximal ideal.

DEFINITION: Let $(P, \leq)$ be a partially ordered set.

(i) A **maximal element** of P is an element $x \in P$ such that for all $y \in P$, one has $x \leq y$ implies $x = y$.

(ii) A **upper bound** for a subset X is an element $x \in P$ such that for all $y \in X$, one has $y \leq x$.

(iii) A subset X of P is a **chain** if for all $x, y \in X$ either $x \leq y$ or $y \leq x$.

ZORN'S LEMMA: Let $(P, \leq)$ be a nonempty partially ordered set. If every chain $C \subseteq P$ has an upper bound $c \in P$, then P has a maximal element.

(3) Zorn's Lemma warmup.

(a) The most common use of Zorn's Lemma occurs in the following situation: $\mathcal{P}(Y)$ is the collection of all subsets of some set Y ordered by inclusion ($A \leq B$ if and only if $A \subseteq B$), and P is some special family of subsets of $\mathcal{P}(Y)$. Rewrite[2] the statement of Zorn's Lemma in this context.

If, P is nonempty and for any nested family of subsets $\{A_\alpha\}_\alpha$ with $A_\alpha \in P$ for all α, there is some $B \in P$ such that $A_\alpha \subseteq B$ for all α, then there is some element $X \in P$ that is not properly contained in any element of P.

(b) In the context above, explain how to use Zorn's lemma to try to show the existence of a *minimal element* of P.

We can consider P as a poset with the alternative partial order $A \leq B$ if and only if $A \supseteq B$. A maximal element of this poset corresponds to a minimal element of P under containment.

(4) Prove Theorem 2.

Fix a ring R. Let S be the set of all proper ideals J in R, which is partially ordered with the inclusion order $\subseteq$. We claim that Zorn's Lemma applies to S. First, S is nonempty, since it contains I. Now consider a chain of proper ideals in R, say $\{J_i\}_i$, all of which contain I. Notice that $J := \bigcup_i J_i$ is an ideal as well (exercise!), and moreover $J \neq R$ since $1 \notin J_i$ for all i.[3] Since each $J_i \supseteq I$, we conclude that $J \supseteq I$. Thus we have checked that $J \in S$. Now this ideal $J \in S$ is an upper bound for our chain $\{J_i\}_i$, and thus Zorn's Lemma applies to S. We conclude that S has a maximal element. Such an element is, by definition, a maximal ideal of R.

(5) Prove or disprove: Any group G has a maximal proper subgroup (meaning a proper subgroup that is maximal among all proper subgroups).

(6) Prove that every prime ideal contains a minimal prime ideal.

[2]Meaning replace all $\leq$ with $\subseteq$ and unpackage the definitions of maximal element and upper bound.