DEFINITION: Let R be a ring. An **ideal** of R (also called a **two-sided ideal**) is a nonempty subset of R such that

(1) I is closed under addition: for all $a, b \in I$, we have $a + b \in I$.
(2) I absorbs multiplication: for all $r \in R$ and $a \in I$, we have $ra \in I$ and $ar \in I$.

A **left ideal** of R is a nonempty subset of R such that

(1) I is closed under addition: for all $a, b \in I$, we have $a + b \in I$.
(2ℓ) I absorbs left multiplication: for all $r \in R$ and $a \in I$, we have $ra \in I$.

The definition of **right ideal** is analogous.

LEMMA 1 (GENERAL RECIPES FOR IDEALS): Let R be a ring.

(i) If I, J are ideals, then $I + J := \{a + b \mid a \in I, b \in J\}$ is an ideal.
(ii) If I, J are ideals, then $IJ := \{\sum_{i=1}^{n} a_i b_i \mid a_i \in I, b_i \in J\}$ is an ideal.
(iii) If $\{I_\alpha\}_{\alpha \in A}$ is an arbitrary collection of ideals of R, then $\bigcap_{\alpha \in A} I_\alpha$ is an ideal.
(iv) If $\{I_\alpha\}_{\alpha \in A}$ is a *chain*[1] of ideals, then $\bigcup_{\alpha \in A} I_\alpha$ is an ideal.
(v) If I is an ideal, then $I[x] := \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in I\}$ is an ideal of $R[x]$.

(1) Working with the definition:

(a) If I is an ideal (or left ideal) of R, explain why $0 \in I$ and $(I, +)$ is a subgroup of $(R, +)$.

Since $I \neq \varnothing$, take $a \in I$, and we have $0 = 0 \cdot a \in I$. To complete the two-step test, we just need to check that I is closed under additive inverses: give $a \in I$, we have $-a = (-1)a \in I$.

(b) Very quickly explain why $\{0\}$ and R are ideals of R. We say that an ideal is **nontrivial** if $I \neq 0$ and proper if $I \neq R$.

$\{0\}$ is closed under addition since $0 + 0 = 0$ and absorbs multiplication since $r \cdot 0 = 0 = 0 \cdot r$ for all $r \in R$. R is closed under addition and absorbs multiplication since addition and multiplication are operations on R.

(c) Explain why an ideal $I \subseteq R$ is proper if and only if $1 \notin I$.

If $1 \notin I$, then clearly $I \neq R$. If $1 \in I$, then for any $r \in R$, $r = 1 \cdot r \in I$.

(d) Quickly explain why "ideal," "left ideal," and "right ideal" are identical notions in a commutative ring.

Absorbs left multiplication is the same as absorbs multiplication (or right multiplication).

[1] This means that for all $\alpha, \beta \in A$, either $I_\alpha \subseteq I_\beta$ or $I_\beta \subseteq I_\alpha$.

(2) Show that the subset

$$\left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \subseteq \mathrm{Mat}_2(\mathbb{R})$$

is a left ideal, but is not a (two-sided) ideal.

It is a left ideal since

$$\begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} ca + db & 0 \\ ea + fb & 0 \end{bmatrix}$$

is in the subset, but not an ideal since, for example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not.

(3) Prove parts (i) and (v) of Lemma 1.

(4) Show[2] that the union of two ideals does not have to be an ideal in general.

(5) Prove parts (ii)–(iv) of Lemma 1.

[2] Hint: Consider $2\mathbb{Z}, 3\mathbb{Z} \subseteq \mathbb{Z}$.

DEFINITION: Let R be a ring. and $S \subseteq R$ be a subset. The **ideal generated by** S is the ideal

$$(S) = \bigcap_{\substack{I \text{ ideal} \\ I \supseteq S}} I.$$

An ideal is **principal** if $I = (a)$ for a single element $a \in R$.

LEMMA 2 (IDEAL GENERATED A SUBSET): Let R be a ring and $S \subseteq R$.

(i) There is an equality $(S) = \{\sum_{i=1}^{n} r_i a_i r_i' \mid r_i, r_i' \in R, a_i \in S\}$.

(ii) If R is commutative, then $(S) = \{\sum_{i=1}^{n} r_i a_i \mid r_i \in R, a_i \in S\}$.

(iii) If R is commutative and $a \in R$, then $(a) = \{ra \mid r \in R\}$.

(6) Let $R = \mathbb{Z}[x]$. Use the Lemma to quickly explain the following:

(a) (2) is the set of all integer polynomials with every coefficient even.

We have $(2) = \{2(a_0 + a_1 x + \cdots + a_n x^n)\} = \{2a_0 + 2a_1 + \cdots + 2a_n x^n\}$.

(b) (x) is the set of all integer polynomials with zero constant term.

We have $(x) = \{x(a_0 + a_1 x + \cdots + a_n x^n)\} = \{a_0 x + \cdots + a_n x^{n+1}\} = \{b_1 x + \cdots + b_n x^n\}$.

(c) $(2, x)$ is the set of all integer polynomials with even constant term.

For $f \in (2, x)$ we can write $f = 2g + xh$; since $2g$ has even constant term and xh has constant term zero, f has even constant term. Conversely, given $f = 2a_0 + a_1 x + \cdots + a_n x^n$, we can write $f = 2a_0 + x(a_1 + \cdots + a_n x^{n-1}) \in (2, x)$.

(7) Show[3] that the ideal $(2, x)$ in $\mathbb{Z}[x]$ is not principal.

(8) Show that if R is noncommutative then one can[4] have $(a) \supsetneq \{rar' \mid r, r' \in R\}$.

[3]Hint: If $(2, x) = (f)$, with $f = a_0 + a_1 x + \cdots + a_n x^n$, note that $2, x \in (f)$. What can you say about a_0?

[4]Hint: You can use the fact that you will prove in HW#11 that $\mathrm{Mat}_n(F)$ has no nontrivial proper ideals if F is a field.