CLASSIFYING ABELIAN GROUPS, AND OTHERS, UP TO ISOMORPHISM

STRUCTURE THEOREM FOR FINITE GENERATED ABELIAN GROUPS: INVARIANT FACTORS: Let G be a finitely generated abelian group. There exist integers $r \geqslant 0$, and $n_i \geqslant 2$, satisfying $n_1 \mid n_2 \mid \cdots \mid n_t$ such that

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_t.$$

Moreover, the list $r, n_1, \ldots, n_t$ is uniquely determined by G.

STRUCTURE THEOREM FOR FINITE GENERATED ABELIAN GROUPS: ELEMENTARY DIVISORS: Let G be a finitely generated abelian group. Then there exist integers $r \geqslant 0$, not necessarily distinct positive prime integers $p_1, \ldots, p_s$, and integers $a_i \geqslant 1$ for $1 \leqslant i \leqslant s$ such that

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/p_1^{a_1} \times \cdots \times \mathbb{Z}/p_s^{a_s}.$$

Moreover, r and s are uniquely determined by G, and the list of prime powers $p_1^{a_1}, \ldots, p_s^{a_s}$ is unique up to the ordering.

(1) Converting between forms:

To convert a cyclic group $\mathbb{Z}/a$ to elementary divisor form, write each $a = p_1^{e_1} \cdots p_s^{e_s}$ as a product of prime powers, and use CRT get

$$\mathbb{Z}/a \cong \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_s^{e_s}.$$

(a) Convert $\mathbb{Z}^2 \times \mathbb{Z}/50 \times \mathbb{Z}/60$ to elementary divisor form.

To convert a group from elementary divisor form to invariant factor form,

- For each distinct prime p_j occurring, take the largest power E_j it has in an elementary divisor, and combine and combine $\prod_j \mathbb{Z}/p_j^{E_j} \cong \mathbb{Z}/(p_1^{E_1} \cdots p_\ell^{E_\ell})$ via CRT. If there's more than one copy of $\mathbb{Z}/p_j^{E_j}$, just take one of the copies and leave the rest.
- Repeat with the remaining factors.

(b) Convert $\mathbb{Z}^3 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/9 \times \mathbb{Z}/27 \times \mathbb{Z}/25$ to invariant factor form.

(2) Which of the following groups of order 2160 are isomorphic or not?

- $\mathbb{Z}/5 \times \mathbb{Z}/12 \times \mathbb{Z}/36$
- $\mathbb{Z}/10 \times \mathbb{Z}/12 \times \mathbb{Z}/18$
- $\mathbb{Z}/30 \times \mathbb{Z}/54$

(3) Classify all *abelian* groups of order 20 up to isomorphism. For each isomorphism class, give its expression in invariant factor form.

(4) Let $p < q$ be primes.

(a) Show that if p does not divide $q-1$, then any group of order pq is isomorphic to C_{pq} by the following steps:

- Use Sylow's Theorem to count the number of Sylow subgroups.
- Apply the Recognition Theorem for direct products.

(b) Show from that if p does divide $q-1$, then there are exactly two groups of order pq up to isomorphism by the following steps:

- Use Sylow's Theorem to count the number of Sylow subgroups.
- Apply the Recognition Theorem for semidirect products.
- Use an Exercise from class about when two semidirect products are isomorphic.

(5) Let p be a prime integer. Let G be a group of order p^2.
  (a) Show[1] that G is abelian.
  (b) Classify all groups of order p^2 up to isomorphism.

(6) Let p, q be primes such that $q = p + 2$ and $p \geq 5$. Show that any group of order p^2q^2 is either isomorphic to a cyclic group or a product of two cyclic groups.

[1] Hint: If not, what can you say about $Z(G)$ and $G/Z(G)$?