THE MAIN THEOREM OF SYLOW THEORY

RECALL: Let G be a finite group and p be a prime number. Write $|G| = p^e m$ with $e \geq 0$ and $p \nmid m$.

- A p-subgroup of G is a subgroup of order p^k for some $k \geq 0$.
- A Sylow p-subgroup of G is a subgroup of order p^e.
- We write $\mathrm{Syl}_p(G)$ for the set of Sylow p-subgroups of G. We often write n_p for $\#\mathrm{Syl}_p(G)$.

MAIN THEOREM OF SYLOW THEORY: Let G be a finite group and p be a prime number. Write $|G| = p^e m$ with $e \geq 0$ and $p \nmid m$.

(1) There exists a Sylow p-subgroup of G.
(2) Every Sylow subgroup is conjugate. Moreover, for any p-subgroup Q and any Sylow p-subgroup P, there is some $g \in G$ such that $Q \leq gPg^{-1}$.
(3) The number of Sylow p-subgroups of G is congruent to 1 modulo p.
(4) The number of Sylow p-subgroups of G divides m.

LEMMA: Let G be a finite group and p be a prime number. Let P be a Sylow p-subgroup of G and Q be any p-subgroup of G. Then $Q \cap N_G(P) = Q \cap P$.

(1) Let $p < q$ be distinct primes and G be a group of order pq. Use the Sylow Theorem to show that G is not simple.

(2) Consider $G = S_4$.
  (a) Show[1] that G has a subgroup isomorphic to D_4, the symmetry group of the square.
  (b) Show that S_4 has exactly three subgroups isomorphic to D_4, that these three are conjugate, and that any subgroup of S_4 of order 8 is isomorphic to D_4.
  (c) Describe the subgroups of order 3 of S_4.

(3) Proof of part (1) of Sylow's Theorem: Fix p. We will argue by induction on n that every group of n has a Sylow p-subgroup.
  (a) Write $n = p^e m$. Address the case $e = 0$. Henceforth assume $e > 0$, so $p \mid n$.
  (b) Case 1: Assume that p divides $|Z(G)|$. Explain why there is some $N \trianglelefteq G$ with $|N| = p$.
  (c) Apply the induction hypothesis to G/N. How can you use this to find a Sylow p-subgroup in G?
  (d) Case 2: Assume that p does not divide $|Z(G)|$. Show that there is some $g \in G$ such that $[G : C_G(g)]$ is *not* a multiple of p and *not* one. What does this say about $|C_G(g)|$? What do you get from the induction hypothesis?

(4) Proof of parts (2) and (3) of Sylow's Theorem: Fix a Sylow p-subgroup P. Let $\mathcal{S}_P$ be the set of conjugates of P, namely $\{gPg^{-1} \mid g \in G\} \subseteq \mathrm{Syl}_p(G)$. We need to show that (2) $\mathrm{Syl}_p(G) = \mathcal{S}_P$ and that (3) $\#\mathrm{Syl}_p(G) \equiv 1 \bmod p$.
  (a) Let Q be any p-subgroup of G, and let Q act on $\mathcal{S}_P$ by conjugation. Use the Lemma to show that for any $P_i \in \mathcal{S}_P$, $\mathrm{Stab}_Q(P_i) = Q \cap P_i$.
  (b) Show that $|\mathcal{S}_P| = \sum_{i=1}^{s} [Q : Q \cap P_i]$ where P_i ranges through a set of representatives of distinct orbits for the action of Q on $\mathcal{S}_P$.
  (c) Take $Q = P$ and WLOG $P_1 = P$. Deduce that $|\mathcal{S}_P| \equiv 1 \bmod p$.
  (d) To show (2) by contradiction, suppose that Q is not contained in any conjugate of P. Observe that $Q \cap P_i \subsetneqq Q$ for all i. Revisit the equation in part (b) and the conclusion of part (c) to obtain a contradiction.
  (e) Deduce part (3) from part (c) and part (2).

[1]Hint: D_4 acts on the vertices of a square.