ORBIT-STABILIZER THEOREM

DEFINITION: Let G be a group acting on a set X, and $x \in X$.

- The **orbit** of x is $\mathrm{Orb}_G(x) = \{g \cdot x \mid g \in G\} \subseteq X$.
- The **stabilizer** of x is $\mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\} \leq G$.

ORBIT-STABILIZER THEOREM: Let G be a group acting on a set X, and $x \in X$. Then

$$|\mathrm{Orb}_G(x)| = [G : \mathrm{Stab}_G(x)].$$

COROLLARY OF ORBIT-STABILIZER THEOREM: Let G be a finite group acting on a set X, and $x \in X$. Then

$$|\mathrm{Orb}_G(x)| \cdot |\mathrm{Stab}_G(x)| = |G|.$$

In particular, the size of any orbit divides the order of G.

(1) Use the Orbit-Stabilizer Theorem and/or its corollary above to quickly explain why the following are *impossible*:

- $S_4 \curvearrowright X$ transitively for a set X with 5 elements.

This would imply that X is a single orbit with 5 elements, but 5 does not divide the order of S_4.

- $G \curvearrowright X$ with $|G| = 16$, $|X|$ odd, and the action has no fixed point[1].

Every orbit has order dividing 16, so is either equal to one (a fixed point) or has an even number of elements. If there are no fixed points, then $|X|$ must be even.

(2) Proof of Theorem/Corollary.

(a) Prove the Orbit-Stabilizer Theorem by showing that the map

$$\{\text{left cosets of } \mathrm{Stab}_G(x) \text{ in } G\} \longrightarrow \mathrm{Orb}_G(x)$$
$$g \cdot \mathrm{Stab}_G(x) \mapsto g \cdot x$$

is a well-defined bijective function.

We have $g\mathrm{Stab}_G(x) = h\mathrm{Stab}_G(x) \Leftrightarrow h^{-1}g \in \mathrm{Stab}_G(x)\mathrm{Stab}_G(x) \Leftrightarrow h^{-1}g \cdot x = x \Leftrightarrow h^{-1} \cdot (g \cdot x) = x \Leftrightarrow g \cdot x = h \cdot x$, so this is well-defined and injective. It is surjective by construction and definition of orbit.

(b) Deduce the Corollary from the Theorem.

Follows from Lagrange.

[1] A **fixed point** of a group action is some $x \in X$ such that $g \cdot x = x$ for all $g \in G$.

(3) Let G be the group of rotational symmetries of a cube.

(a) Explain very briefly why G acts on the set F of faces of the cube.

Any symmetry sends faces to other faces.

(b) Explain why $G \curvearrowright F$ is transitive.

There is a rotation that takes any face to any other face.

(c) Compute $\mathrm{Stab}_G(f)$ for $f \in F$.

If the top face stays on top, there are only four rotations.

(d) Compute $|G|$.

By Orbit-Stabilizer, there are $6 \cdot 4 = 24$ elements.

(4) Let G be the group of rotational symmetries of a cube.
   (a) Explain briefly why G acts on the set of long diagonals D (line segments between pairs of opposite vertices) of the cube.
   (b) Explain why, if we know that $G \curvearrowright D$ is faithful, then $G \cong S_4$.
   (c) Show that $G \curvearrowright D$ is faithful.

(5) For the other platonic solids, compute the order of the rotational symmetry group. Can you compute the rotational symmetry group up to isomorphism as a group we already know?