DEFINITION: Let G be a group acting on a set X, and  $x \in X$ .

- The **orbit** of x is  $Orb_G(x) = \{g \cdot x \mid g \in G\} \subseteq X$ .
- The stabilizer of x is  $\operatorname{Stab}_G(x) = \{g \in G \mid g \cdot x = x\} \leq G$ .

ORBIT-STABILIZER THEOREM: Let G be a group acting on a set X, and  $x \in X$ . Then

$$|\operatorname{Orb}_G(x)| = [G : \operatorname{Stab}_G(x)].$$

COROLLARY OF ORBIT-STABILIZER THEOREM: Let G be a finite group acting on a set X, and  $x \in X$ . Then

$$|\operatorname{Orb}_G(x)| \cdot |\operatorname{Stab}_G(x)| = |G|.$$

In particular, the size of any orbit divides the order of G.

- **(1)** Use the Orbit-Stabilizer Theorem and/or its corollary above to quickly explain why the following are *impossible*:
  - $S_4 \curvearrowright X$  transitively for a set X with 5 elements.
  - $G \curvearrowright X$  with |G| = 16, |X| odd, and the action has no fixed point<sup>1</sup>.
- **(2)** Proof of Theorem/Corollary.
  - (a) Prove the Orbit-Stabilizer Theorem by showing that the map

{left cosets of 
$$\operatorname{Stab}_G(x)$$
 in  $G$ }  $\longrightarrow \operatorname{Orb}_G(x)$   
 $g \cdot \operatorname{Stab}_G(x) \mapsto g \cdot x$ 

is a well-defined bijective function.

- **(b)** Deduce the Corollary from the Theorem.
- (3) Let G be the group of rotational symmetries of a cube.
  - (a) Explain very briefly why G acts on the set F of faces of the cube.
  - **(b)** Explain why  $G \curvearrowright F$  is transitive.
  - (c) Compute  $\operatorname{Stab}_G(f)$  for  $f \in F$ .
  - (d) Compute |G|.
- (4) Let G be the group of rotational symmetries of a cube.
  - (a) Explain briefly why G acts on the set of long diagonals D (line segments between pairs of opposite vertices) of the cube.
  - (b) Explain why, if we know that  $G \curvearrowright D$  is faithful, then  $G \cong S_4$ .
  - (c) Show that  $G \curvearrowright D$  is faithful.
- (5) For the other platonic solids, compute the order of the rotational symmetry group. Can you compute the rotational symmetry group up to isomorphism as a group we already know?

<sup>&</sup>lt;sup>1</sup>A **fixed point** of a group action is some  $x \in X$  such that  $g \cdot x = x$  for all  $g \in G$ .