THE ISOMORPHISM THEOREMS

0. UNIVERSAL MAPPING PROPERTY FOR QUOTIENT GROUPS: Let $f : G \to H$ be a group homomorphism. If $N \trianglelefteq G$ and $N \subseteq \ker(f)$, then there is a unique group homomorphism $\overline{f} : G/N \to H$ such that $f = \overline{f} \circ \pi$; i.e., the diagram

$$\begin{tikzcd} & G \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ G/N \arrow[rr, dashed, "\overline{f}"'] & & H \end{tikzcd}$$

commutes. Moreover, $\operatorname{im}(\overline{f}) = \operatorname{im}(f)$, and $\ker(\overline{f}) = \{gN \mid g \in \ker(f)\}$.

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1. FIRST ISOMORPHISM THEOREM: Let $f : G \to H$ be a group homomorphism. Then

$$\begin{array}{ccc} G/\ker(f) & \xrightarrow{\overline{f}} & \operatorname{im}(f) \\ g \cdot \ker(f) & \mapsto & f(g) \end{array}$$

is an isomorphism.

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2. DIAMOND ISOMORPHISM THEOREM: Let G be a group, $H \leq G$, and $N \trianglelefteq G$. Then

$$HN \leq G, \quad N \cap H \trianglelefteq H, \quad N \trianglelefteq HN$$

and

$$\begin{array}{ccc} \dfrac{H}{N \cap H} & \longrightarrow & \dfrac{HN}{N} \\ h(N \cap H) & \mapsto & hN \end{array}$$

is an isomorphism.

$$\begin{tikzcd} & G \arrow[ddl, dash, "\leq"'] \arrow[d, dash, "\leq"] \arrow[ddr, dash, "\trianglelefteq"] & \\ & HN \arrow[dl, dash, "\leq"] \arrow[dr, dash, "\trianglelefteq"'] & \\ H \arrow[dr, dash, "\trianglelefteq"'] & & N \arrow[dl, dash, "\leq"] \\ & N \cap H & \end{tikzcd}$$

3. CANCELLING ISOMORPHISM THEOREM: Let G be a group, $M \le N \le G$, $M \trianglelefteq G$, and $N \trianglelefteq G$. Then

$$M \trianglelefteq N, \qquad N/M \trianglelefteq G/M,$$

and the map

$$\begin{array}{ccc} \dfrac{(G/M)}{(N/M)} & \longrightarrow & G/N \\ gM(N/M) & \mapsto & gN \end{array}$$

is an isomorphism.

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4. LATTICE ISOMORPHISM THEOREM: Let G be a group and N a normal subgroup of G, and let $\pi \colon G \twoheadrightarrow G/N$ be the quotient map. There is an order-preserving bijection of posets (a lattice isomorphism)

$$\begin{tikzcd} \{\text{subgroups of } G \text{ that contain } N\} \arrow[r, shift left, "\Psi"] & \{\text{subgroups of } G/N\} \arrow[l, shift left, "\Phi"] \\ H \arrow[r, mapsto] & \pi(H) = H/N \\ \pi^{-1}(A) = \{x \in G \mid \pi(x) \in A\} & A \arrow[l, mapsto] \end{tikzcd}$$

Then this bijection enjoys the following properties:

(1) Subgroups correspond to subgroups:

$$H \le G \iff H/N \le G/N.$$

(2) Normal subgroups correspond to normal subgroups:

$$H \trianglelefteq G \iff H/N \trianglelefteq G/N.$$

(3) Indices are preserved:

$$[G : H] = [G/N : H/N].$$

(4) Intersections and generated subgroups are preserved:

$$H/N \cap K/N = (H \cap K)/N \quad \text{and} \quad \langle H/N \cup K/N \rangle = \langle H \cup K \rangle/N.$$