Normal subgroups

Definition: A subgroup N of a group G is **normal** if $gNg^{-1} = N$ for all $g \in G$, where $gNg^{-1} = \{gng^{-1} \mid n \in N\}$. We write $N \trianglelefteq G$ to indicate that N is a normal subgroup of G.

Lemma: Let N be a subgroup of a group G. The following are equivalent:

(1) N is a normal subgroup of G.
(2) For all $g \in G$, $gNg^{-1} \subseteq N$.
(3) For all $g \in G$, the *left coset* gN is equal to the *right coset* Ng.
(4) For all $g \in G$, $gN \subseteq Ng$.
(5) For all $g \in G$, $Ng \subseteq gN$.

(1) Examples of normal subgroups: Use the definition and/or the Lemma to show the following:

(a) If G is an abelian group and $H \leq G$, then $H \trianglelefteq G$.

Let $h \in H$ and $g \in G$. Since G is abelian, we have $ghg^{-1} = gg^{-1}h = h$. Thus, $gHg^{-1} \subseteq H$, so H is normal.

(b) The center $Z(G)$ of a group G is a normal subgroup[1] of G.

Let $z \in Z(G)$ and $g \in G$. Since z is in the center, we have $gzg^{-1} = gg^{-1}z = z$. Thus, $gZ(G)g^{-1} \subseteq Z(G)$, so $Z(G)$ is normal.

(c) The[2] group $K = \{e, (12)(34), (13)(24), (14)(23)\} \leq S_4$ is normal.

First, we should check that it is indeed a subgroup. To see it, we can just multiply out elements and check that the result is in K. For each product involving e, there is nothing to check, and each element besides e has order 2, so its product with itself is in K. We then just verify

$$(12)(34)(13)(24) = (13)(24)(12)(34) = (14)(23), \; (12)(34)(14)(23) = (14)(23)(12)(34) = (13)(24), \text{ and } (13)(2$$

Note also that K is abelian. Now we check that K is normal in G. For any τ in G, using the exercise from the homework, if $(i\,j)(k\,\ell)$ is a product of two disjoint transpositions, then

$$\tau(i\,j)(k\,\ell)\tau^{-1} = \tau(i\,j)\tau^{-1}\tau(k\,\ell)\tau^{-1} = (\tau(i)\,\tau(j))(\tau(k)\,\tau(\ell))$$

is as well, and is thus an element of K. This shows that K is normal.

(d) Let $H = \{e, (12)(34)\} \leq K$, with K as above. Check that $H \trianglelefteq K$ and $K \trianglelefteq S_4$, but $H \ntrianglelefteq S_4$. Draw a moral from this example.

Since K is abelian, $H \trianglelefteq K$. However, H is not a normal subgroup of S_4, since conjugating $(12)(34)$ by $(1\,3)$ yields $(1\,4)(2\,3) \notin H$. Normal subgroup is not a transitive relation.

(e) Is the subgroup of all rotations a normal subgroup of D_n?

[1] Recall that we have already shown that $Z(G) \leq G$.

[2] Hint: Recall from HW 1 that $\tau(i\;j)\tau^{-1} = (\tau(i)\;\tau(j))$.

Yes.

(f) Is the subgroup generated by one reflection a normal subgroup of D_n?

No.

(2) Prove the Lemma.

(3) Let G be a group and $H \leq G$ a subgroup of index 2. Show that H must be normal.

RECALL:

- An equivalence relation $\sim$ on a group is **compatible with multiplication** if $x \sim y$ implies $xz \sim yz$ and $zx \sim zy$ for all $x, y, z \in G$. If $\sim$ is compatible with multiplication, then the equivalence classes of $\sim$ obtain a well-defined group structure via the rule $[x][y] = [xy]$.
- For a subgroup H, we define an equivalence relation on G by $x \sim_H y$ if and only $hx = y$ for some $h \in H$. The equivalence classes are the right cosets Hx.

THEOREM: Let G be a group. An equivalence relation $\sim$ is compatible with multiplication if and only if $\sim = \sim_N$ for some $N \trianglelefteq G$.

COROLLARY: If G is a group and N is a normal subgroup, the collection of left cosets $\{gN \mid g \in G\}$ of N forms a group by the rule $gN \cdot hN = ghN$.

(4) Explain why the Corollary follows from the Theorem.

By the Theorem, if N is normal, the equivalence relation $\sim_N$ is compatible with multiplication, and thus by the recollection above, we get an induced group structure on the equivalence classes. The equivalence classes of $\sim_N$ are the right cosets of N in G; since N is normal, we can equivalently consider these as the left cosets of N in G. The rule for the group action is the same as in the recollection just using the concrete notation gN for the equivalence class $[g]$.

(5) Prove the $(\Leftarrow)$ direction of the Theorem.

Suppose that N is normal, and take $\sim_N$. Let $x, y, z \in G$. If $x \sim_N y$, then $Nx = Ny$, so $Nxz = Nyz$, and hence $xz \sim_N yz$. But we also have $xN = yN$, so $zxN = zyN$, so $Nzx = Nzy$ and $zx \sim_N zy$.

(6) Prove[3] the $(\Rightarrow)$ direction of the Theorem.

[3]Hint: The main issue here is to find a candidate N. Think first about how you would reconstruct N from $\sim_N$.