Normal subgroups

Definition: A subgroup N of a group G is **normal** if $gNg^{-1} = N$ for all $g \in G$, where $gNg^{-1} = \{gng^{-1} \mid n \in N\}$. We write $N \trianglelefteq G$ to indicate that N is a normal subgroup of G.

Lemma: Let N be a subgroup of a group G. The following are equivalent:

(1) N is a normal subgroup of G.
(2) For all $g \in G$, $gNg^{-1} \subseteq N$.
(3) For all $g \in G$, the *left coset* gN is equal to the *right coset* Ng.
(4) For all $g \in G$, $gN \subseteq Ng$.
(5) For all $g \in G$, $Ng \subseteq gN$.

(1) Examples of normal subgroups: Use the definition and/or the Lemma to show the following:
  (a) If G is an abelian group and $H \leq G$, then $H \trianglelefteq G$.

  (b) The center $Z(G)$ of a group G is a normal subgroup[1] of G.
  (c) The[2] group $K = \{e, (12)(34), (13)(24), (14)(23)\} \leq S_4$ is normal.
  (d) Let $H = \{e, (12)(34)\} \leq K$, with K as above. Check that $H \trianglelefteq K$ and $K \trianglelefteq S_4$, but $H \ntrianglelefteq S_4$. Draw a moral from this example.
  (e) Is the subgroup of all rotations a normal subgroup of D_n?
  (f) Is the subgroup generated by one reflection a normal subgroup of D_n?

(2) Prove the Lemma.

(3) Let G be a group and $H \leq G$ a subgroup of index 2. Show that H must be normal.

Recall:

- An equivalence relation $\sim$ on a group is **compatible with multiplication** if $x \sim y$ implies $xz \sim yz$ and $zx \sim zy$ for all $x, y, z \in G$. If $\sim$ is compatible with multiplication, then the equivalence classes of $\sim$ obtain a well-defined group structure via the rule $[x][y] = [xy]$.
- For a subgroup H, we define an equivalence relation on G by $x \sim_H y$ if and only $hx = y$ for some $h \in H$. The equivalence classes are the right cosets Hx.

Theorem: Let G be a group. An equivalence relation $\sim$ is compatible with multiplication if and only if $\sim = \sim_N$ for some $N \trianglelefteq G$.

Corollary: If G is a group and N is a normal subgroup, the collection of left cosets $\{gN \mid g \in G\}$ of N forms a group by the rule $gN \cdot hN = ghN$.

(4) Explain why the Corollary follows from the Theorem.

(5) Prove the $(\Leftarrow)$ direction of the Theorem.

[1] Recall that we have already shown that $Z(G) \leq G$.
[2] Hint: Recall from HW 1 that $\tau(i\ j)\tau^{-1} = (\tau(i)\ \tau(j))$.

(6) Prove[3] the $(\Rightarrow)$ direction of the Theorem.

[3]Hint: The main issue here is to find a candidate N. Think first about how you would reconstruct N from $\sim_N$.