UNIVERSAL MAPPING THEOREM FOR CYCLIC GROUPS: Let $G = \langle x \rangle$ be a cyclic group and H be an arbitrary group.

(1) If $|x| = n < \infty$ and $y \in H$ is such that $y^n = e$, then there is a unique homomorphism $f : G \to H$ such that $f(x) = y$.
(2) If $|x| = \infty$ and $y \in H$ is arbitrary, then there is a unique homomorphism $f : G \to H$ such that $f(x) = y$.

DEFINITION:

- The **infinite cyclic group** is the group $C_\infty = \{a^j \mid j \in \mathbb{Z}\}$ with operation $a^j a^k = a^{j+k}$. Its presentation[1]is $\langle a \mid \varnothing \rangle$.
- For any $n \in \mathbb{Z}_{\geq 1}$, the cyclic group of order n is the group $C_n = \{a^j \mid j \in \{0, 1, \dots, n-1\}\}$ with operation $a^j a^k = a^{j+k \pmod n}$. Its presentation is $\langle a \mid a^n = e \rangle$.

CLASSIFICATION OF CYCLIC GROUPS: Every infinite cyclic group is isomorphic to C_∞. Every cyclic group of order n is isomorphic to C_n.

(1) Use the Universal Mapping Theorem for cyclic groups to prove the classification of cyclic groups.

Let $G = \langle x \rangle$ be an infinite cyclic group. By the UMP for cyclic groups, there is a homomorphism $f : G \to C_\infty$ mapping $x \mapsto a$. Conversely, by the UMP for cyclic groups, there is a homomorphism $g : C_\infty \to G$ mapping $a \mapsto x$. The composition $fg : C_\infty \to C_\infty$ maps $a \mapsto a$; the identity map is another such homomorphism, so by the uniqueness part of the UMP, fg is the identity on C_∞. For the same reason, $gf : G \to G$ is the identity. It follows that f is an isomorphism.

Let $G = \langle x \rangle$ be a cyclic group of order n. Since $a \in C_n$ has order n, there is a homomorphism $f : G \to C_n$ mapping $x \mapsto a$. Likewise, there is a homomorphism $g : C_n \to G$ by the UMP. Following the same argument as above, we see that these are mutually inverse, so f is an isomorphism.

(2) Prove the Universal mapping theorem for cyclic groups.

We know that homomorphisms are uniquely determined by their images on a generating set, so in each case we just need to show existence.

In either case, define $f(x^i) = y^i$. We must show this function is a well-defined group homomorphism. To see that f is well-defined, suppose $x^i = x^j$ for some $i, j \in \mathbb{Z}$. Then, since $x^{i-j} = e_G$, using earlier work, we have

$$\begin{cases} n \mid i-j & \text{if } |x| = n \\ i-j = 0 & \text{if } |x| = \infty \end{cases} \implies \begin{cases} y^{i-j} = y^{nk} & \text{if } |x| = n \\ y^{i-j} = y^0 & \text{if } |x| = \infty \end{cases} \implies y^{i-j} = e_H \implies y^i = y^j.$$

Thus, if $x^i = x^j$ then $f(x^i) = y^i = y^j = f(x^j)$. In particular, if $x^k = e$, then $f(x^k) = e$, and f is well-defined.

The fact that f is a homomorphism is immediate:

$$f(x^i x^j) = f(x^{i+j}) = y^{i+j} = y^i y^j = f(x^i) f(x^j).$$

(3) Classify all subgroups of C_∞ and describe the subgroup lattice.

[1]We write the empty set in the relations spot to indicate that there are no defining relations.