DEFINITION: Let G be a group and X be a set. A **group action** of G on X is a function $G \times X \to X$ typically written as $(g, x) \mapsto g \cdot x$ such that

(1) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$, and
(2) $e_G \cdot x = x$ for all $x \in X$.

Given a group action of G on X and $x \in X$, the **orbit** of x is

$$\mathrm{Orb}_G(x) := \{g \cdot x \mid g \in G\}.$$

LEMMA: Given a group action of G on X,

- for $x, y \in X$, either $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(y)$ or $\mathrm{Orb}_G(x) \cap \mathrm{Orb}_G(y) = \varnothing$.
- $X = \bigcup_{x \in X} \mathrm{Orb}_G(x)$.

DEFINITION: A group action of G on X is

- **transitive** if $\mathrm{Orb}_G(x) = X$ for some $x \in X$.
- **faithful** if $g \cdot x = x$ for all $x \in X$ implies that $g = e$.

(1) Let G be a group acting on a set X. For $x, y \in X$, write $x \sim y$ if there exists $g \in G$ such that $g \cdot x = y$.

(a) Show that $\sim$ is an equivalence relation[1].

Since $e \cdot x = x$, we have $x \sim x$, so $\sim$ is reflexive. If $x \sim y$, then $g \cdot x = y$ for some $g \in G$; then $g^{-1} \cdot y = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x$, so $y \sim x$; hence $\sim$ is symmetric. If $x \sim y$ and $y \sim z$, then we have $g \cdot x = y$ and $h \cdot y = z$ for some $g, h \in G$. Then $(hg) \cdot x = h \cdot (g \cdot x) = h \cdot y = z$, so $x \sim z$. This shows that $\sim$ is transitive.

(b) Relate the previous part to the Lemma.

If $\sim$ is an equivalence relation on X, the equivalence classes form a partition of X. The conclusion of the Lemma is saying that the equivalnce classes (orbits) are a partition.

(c) Suppose that X is a finite set, and $X_1, \dots, X_\ell$ are the distinct orbits of G acting on X. Explain:

$$|X| = \sum_{i=1}^{\ell} |X_i|.$$

This follows immediately from the Lemma.

(2) Dihedral group actions: Let D_n be the group of symmetries of a regular n-gon P_n in $\mathbb{R}^2$.

(a) Explain why/how D_n acts naturally on P_n. Is this action transitive? Is it faithful?

By definition elements of D_n are functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(P_n) \subseteq P_n$, so we may consider $f \cdot x = f(x)$ for $f \in D_n$ and $x \in P_n$. The identity e of D_n is the identity function on P_n, so $e \cdot x = x$ for all $x \in P_n$. The operation in D_n is composition of

[1] Recall that a relation on a set is an **equivalence relation** if it is *reflexive*, *symmetric*, and *transitive*.

functions, so for $g, h \in D_n$, $(gh) \cdot x = g(h(x)) = g \cdot (h \cdot x)$. This verifies that this is an action. It is not transitive, since the center of P_n cannot be moved to a vertex of P_n, for example. It is faithful, since an isometry that fixes every point of P_n must be the identity element of P_n.

(b) Explain why/how D_n acts naturally on the set of vertices of P_n. Is this action transitive? Is it faithful?

The action of D_n on P_n restricts to an action on the set of vertices: this is because we proved that every isometry of P_n sends vertices to vertices. This action is now transitive, as we can send any vertex to any other (e.g., by a rotation). It is still faithful.

(3) Group actions on $X \longleftrightarrow$ homomorphisms to $\mathrm{Perm}(X)$:

(a) Let G be a group acting on a set X. For $g \in G$, let $\mu_g : X \to X$ be the function $\mu_g(x) = g \cdot x$, which we made out of the group action. Consider the function

$$\begin{aligned} \rho : G &\to \mathrm{Perm}(X) \\ g &\mapsto \mu_g \end{aligned}$$

Show that ρ is a group homomorphism[2]. We call ρ the **permutation representation** associated to the given group action.

We claim that $\mu_g \circ \mu_h = \mu_{gh}$. Indeed, for any $x \in X$, we have $\mu_g \mu_h(x) = \mu_g(h \cdot x) = g \cdot (h \cdot x) = (gh) \cdot x = \mu_{gh}(x)$. In particular, $\mu_g \circ \mu_{g^{-1}} = \mu_e = \mu_{g^{-1}} \circ \mu_g$, and μ_e is the identity function on X (by the corresponding group action axiom). In particular, μ_g is invertible as a function, and hence is a permutation of X.
By the computation above, we have $\rho(g) \circ \rho(h) = \mu_g \circ \mu_h = \mu_{gh} = \rho(gh)$ for all $g, h \in G$, so ρ is a group homomorphism.

(b) Label the vertices of a square counterclockwise by $\{1, 2, 3, 4\}$. Write out the induced homomorphism $D_4 \to S_4$ coming from the action of D_4 on the vertices as in (2.b) above.

It suffices to compute the images of our generators r, s, for a reflection s, e.g., the one over the line through 1 and 3. Since r sends vertices $1, 2, 3, 4$ to $2, 3, 4, 1$ respectively, the corresponding permutation is $(1\,2\,3\,4)$. Since s sends vertices $1, 2, 3, 4$ to $1, 4, 3, 2$ respectively, the corresponding permutation is $(2\,4)$.

(c) Let G be a group, X a set, and $\rho : G \to \mathrm{Perm}(X)$ a group homomorphism. Give a natural recipe for a group action of G on X, and verify that this is indeed a group action.

We can set $g \cot x = \rho(g)(x)$. Let us verify the axioms. We have $\rho(e)$ is the identity of $\mathrm{Perm}(X)$, so $\rho(e)(x) = x$ for all $x \in X$, and thus $e \cdot x = x$ for all $x \in X$. Given $g, h \in G$, $\rho(gh) = \rho(g)\rho(h)$, so $\rho(gh)(x) = \rho(g)\rho(h)(x) = \rho(g)(\rho(h)(x))$ for all $x \in X$. Thus, $(gh) \cdot x = g \cdot (h \cdot x)$ for all $x \in X$.

[2] Warning: you should also show that μ_g is actually an element of $\mathrm{Perm}(X)$. One good way to do this is to show that $\mu_{g^{-1}}$ is the inverse function of μ_g.

(4) Let G be a group acting on a set X. Complete the following sentence, and prove your answer:
The action of G on X is faithful if and only if the associated permutation representation $\rho : G \to \mathrm{Perm}(X)$ is _____________.

(5) Linear representations on $K^n \longleftrightarrow$ homomorphisms to $\mathrm{GL}_n(K)$:
   (a) Let G be a group and K be a field (you can assume $K = \mathbb{R}$ if you want.) A **linear action** of G on K^n is a group action of G on K^n such that for each $g \in G$, the function $\mu_g : K^n \to K^n$ as in (3) is a linear transformation over K. Given a linear action of G on K^n, show that there is natural group homomorphism $\rho : G \to \mathrm{GL}_n(K)$.
   (b) Conversely, given a group homomorphism $\rho : G \to \mathrm{GL}_n(K)$, give a natural recipe for a linear action of G on K^n.