GROUP ACTIONS

DEFINITION: Let G be a group and X be a set. A **group action** of G on X is a function $G \times X \to X$ typically written as $(g, x) \mapsto g \cdot x$ such that

(1) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$, and
(2) $e_G \cdot x = x$ for all $x \in X$.

Given a group action of G on X and $x \in X$, the **orbit** of x is

$$\mathrm{Orb}_G(x) := \{g \cdot x \mid g \in G\}.$$

LEMMA: Given a group action of G on X,

- for $x, y \in X$, either $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(y)$ or $\mathrm{Orb}_G(x) \cap \mathrm{Orb}_G(y) = \varnothing$.
- $X = \bigcup_{x \in X} \mathrm{Orb}_G(x)$.

DEFINITION: A group action of G on X is

- **transitive** if $\mathrm{Orb}_G(x) = X$ for some $x \in X$.
- **faithful** if $g \cdot x = x$ for all $x \in X$ implies that $g = e$.

(1) Let G be a group acting on a set X. For $x, y \in X$, write $x \sim y$ if there exists $g \in G$ such that $g \cdot x = y$.
  (a) Show that $\sim$ is an equivalence relation[1].
  (b) Relate the previous part to the Lemma.
  (c) Suppose that X is a finite set, and $X_1, \dots, X_\ell$ are the distinct orbits of G acting on X. Explain:

$$|X| = \sum_{i=1}^{\ell} |X_i|.$$

(2) Dihedral group actions: Let D_n be the group of symmetries of a regular n-gon P_n in $\mathbb{R}^2$.
  (a) Explain why/how D_n acts naturally on P_n. Is this action transitive? Is it faithful?
  (b) Explain why/how D_n acts naturally on the set of vertices of P_n. Is this action transitive? Is it faithful?

(3) Group actions on $X \longleftrightarrow$ homomorphisms to $\mathrm{Perm}(X)$:
  (a) Let G be a group acting on a set X. For $g \in G$, let $\mu_g : X \to X$ be the function $\mu_g(x) = g \cdot x$, which we made out of the group action. Consider the function

$$\begin{aligned} \rho : G &\to \mathrm{Perm}(X) \\ g &\mapsto \mu_g \end{aligned}$$

  Show that ρ is a group homomorphism[2]. We call ρ the **permutation representation** associated to the given group action.
  (b) Label the vertices of a square counterclockwise by $\{1, 2, 3, 4\}$. Write out the induced homomorphism $D_4 \to S_4$ coming from the action of D_4 on the vertices as in (2.b) above.

[1]Recall that a relation on a set is an **equivalence relation** if it is *reflexive*, *symmetric*, and *transitive*.

[2]Warning: you should also show that μ_g is actually an element of $\mathrm{Perm}(X)$. One good way to do this is to show that $\mu_{g^{-1}}$ is the inverse function of μ_g.

(c) Let G be a group, X a set, and $\rho : G \to \mathrm{Perm}(X)$ a group homomorphism. Give a natural recipe for a group action of G on X, and verify that this is indeed a group action.

(4) Let G be a group acting on a set X. Complete the following sentence, and prove your answer: The action of G on X is faithful if and only if the associated permutation representation $\rho : G \to \mathrm{Perm}(X)$ is ______________.

(5) Linear representations on $K^n \longleftrightarrow$ homomorphisms to $\mathrm{GL}_n(K)$:

(a) Let G be a group and K be a field (you can assume $K = \mathbb{R}$ if you want.) A **linear action** of G on K^n is a group action of G on K^n such that for each $g \in G$, the function $\mu_g : K^n \to K^n$ as in (3) is a linear transformation over K. Given a linear action of G on K^n, show that there is natural group homomorphism $\rho : G \to \mathrm{GL}_n(K)$.

(b) Conversely, given a group homomorphism $\rho : G \to \mathrm{GL}_n(K)$, give a natural recipe for a linear action of G on K^n.