Problem Set 6.5

Not to be turned in for credit

Problem 1. (a) Let G be a group of order 15 acting on a set S with 7 elements. Prove that there exists at least one fixed point, i.e., there exists $s \in S$ such that $g \cdot s = s$ for all $g \in G$.

(b) Give an example of an action of C_{15} on a set with 8 elements with no fixed points. Justify.

Problem 2. Let G be a group acting on a set A, and let H be a subgroup of G satisfying the condition that the induced action of H on A is transitive (that is, for all $a, b \in A$ there is an $h \in H$ with $ha = b$). Let $t \in A$, and let $\operatorname{Stab}_G(t)$ be the stabilizer of t in G. Show that $G = H \operatorname{Stab}_G(t)$.

Problem 3. Let G be a finite group and let H be a proper subgroup of G with $[G : H] = h$.

1. Prove that H has at most h distinct conjugate sets gHg^{-1} for $g \in G$.
2. Prove that $G \neq \bigcup_{g \in G} gHg^{-1}$.

Problem 4. Let G be a group of order p^n, for some prime p, acting on a finite set X.

(a) Suppose p does not divide $\#X$. Prove that there exists some element of X that is fixed by all elements of G.

(b) **Prove or Disprove:** Suppose G acts faithfully on X. (Recall this means that if $g \cdot x = x$ for all $x \in X$, then $g = e_G$.) Prove that $\#X \geq n \cdot p$.