Problem Set 12

Due Thursday, December 4

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Problem 1. Let $I = (2, x)$ in $R = \mathbb{Z}[x]$.

(a) Show that $\mathfrak{m} = (2, x)$ is a maximal ideal.

(b) Show that $(2, x)$ is not a principal ideal.

Problem 2. Let I and J be ideals of a commutative ring R with $1 \neq 0$. You can use without proof that $I + J$, $I \cap J$, and IJ are ideals of R.

(a) Show that $IJ \subseteq I \cap J$.

(b) Give an example where $IJ \neq I \cap J$.

(c) Suppose that $I + J = R$. Show that $IJ = I \cap J$.

(d) Suppose m and n are distinct maximal ideals of a commutative ring R. Prove that $mn = m \cap n$.

Hint: First consider $m + n$.

(e) Suppose that $I + J = R$. Show that there is a ring isomorphism $R/(I \cap J) \cong R/I \times R/J$.

Problem 3. Let R be a commutative ring. Prove[1] that the set of prime ideals of R has a minimal element with respect to inclusion.

For the remaining problem, you can use the following theorem, to be covered next Monday.

THEOREM: Let R be a commutative ring, and $g = a_n x^n + \cdots + a_1 x + a_0$ a polynomial in $R[x]$ with a_n a unit in R. Then for any $f \in R[x]$, there exists a unique pair of polynomials $q, r \in R[x]$ such that

- $f = qg + r$, and
- $r = 0$ or $\deg(r) < \deg(g)$.

DEFINITION: Let R be a commutative ring, and $f \in R[x]$ a polynomial. We say that $r \in R$ is a **root** of f if $\mathrm{ev}_r(f) = 0$.

[1]Note: (0) is not prime unless R is a domain.

Problem 4. Let R be a commutative ring and $f \in R[x]$.

(a) Show that if $r \in R$ is a root of f, then f is a multiple of the polynomial $x - r$ in $R[x]$.

(b) Show that if R is an integral domain and $\deg(f) = d$, then f has at most d roots.

(c) Give an example of a polynomial f over a commutative ring R that has more than $\deg(f)$ roots in R.