Problem Set 1
Due Wednesday, September 3

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Problem 1. Let G be a group and $x \in G$ any element. Recall that $|x|$ denotes the *order* of x, defined to be the least integer $n \geqslant 1$ such that $x^n = e$; if no such integer exists, we say $|x| = \infty$. Also, let $|G|$ denote the cardinality of G; note that $|G|$ is an element of $\{1, 2, 3, \ldots\} \cup \{\infty\}$.

(a) Prove that if $|x| = n$, then $e, x, \ldots, x^{n-1}$ are all distinct elements of G.

(b) Prove that if $|x| = \infty$, then $x^i \neq x^j$ for all positive integers $i \neq j$.

(c) Conclude[1] that $|x| \leqslant |G|$ in all cases.

Problem 2. A group G is called *cyclic* if it is generated by a single element.

(a) Prove[2] that any cyclic group is abelian.

(b) Prove that $(\mathbb{Q}, +)$ is not a cyclic group.

(c) Prove that $\mathrm{GL}_2(\mathbb{Z}/2)$ is not cyclic.

Problem 3. Let $n \geq 2$, and consider[3] the symmetric group S_n.

(a) Let $\tau \in S_n$ be a permutation, and $(i_1 \, i_2 \, \cdots \, i_k)$ be a k-cycle. Show that

$$\tau(i_1 \, i_2 \, \cdots \, i_k)\tau^{-1} = (\tau(i_1) \, \tau(i_2) \, \cdots \, \tau(i_k)).$$

(b) Show that S_n is generated by (12) and the n-cycle $(12 \cdots n)$.

(c) Show that, if $n \geq 3$, then $Z(S_n) = \{e\}$.

Problem 4. (a) Suppose the cycle type of $\sigma \in S_n$ is $m_1, m_2, \ldots, m_k$. Recall this means that σ a product of disjoint cycles of lengths $m_1, m_2, \ldots, m_k$. Prove that $|\sigma| = \mathrm{lcm}(m_1, \ldots, m_k)$.

(b) Given an example of two permutations $\sigma, \tau \in S_n$ such that $|\sigma\tau| > \mathrm{lcm}(|\sigma|, |\tau|)$.

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[1] We will soon prove Lagrange's Theorem, which states that $|x|$ divides $|G|$ when G is finite. For now, use the definitions and not this result that we have not yet established.

[2] Your proof will be very short, as you can use the fact that $x^i x^j = x^{i+j}$ without proof.

[3] Note: If you are unsure which formulas about permutations require proof, please ask.