Problem Set 7

Due Friday, October 24

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly.* As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Problem 1. Let p be a prime number and $|G| = p^e$ for some $e \geq 1$.

(a) Show that for any nontrivial normal subgroup $N \trianglelefteq G$, we have $N \cap Z(G) \neq \{e\}$.

(b) Show that for every $m \leq e$, there is a subgroup of G of order p^m.

Problem 2. Show that if G is a nonabelian group of order 21, then there is only on possible class equation for G, meaning that the numbers appearing are uniquely determined up to order.

Problem 3. Let G be a a finite group acting transitively on a set X.

(a) Show that for any $x, y \in X$, the stabilizer subgroups $\mathrm{Stab}_G(x)$ and $\mathrm{Stab}_G(y)$ are conjugate subgroups.

(b) Show[1] that if $|X| > 1$, then there exists some $g \in G$ such that $g \cdot z \neq z$ for all $z \in X$.

[1] Hint: Use a Theorem from class/notes.