Final Exam

Instructions: Solve *two* problems from Part 1 and *two* problems from Part 2. You may use any results proved in class or in the problem sets, except for the specific question being asked. You should clearly state any facts you are using. You are also allowed to use anything stated in one problem to solve a different problem, even if you have not yet proved it. Remember to show all your work, and to write clearly and using complete sentences. No calculators, notes, cellphones, smartwatches, or other outside assistance allowed.

Part 1: Groups

Choose *two* of the following problems.

(1) (a) Show that there exists a nonabelian group of order 27.

(b) Give, with justification, a presentation for the group you found in part (a).

(2) Prove that no group of order $224 = 2^5 \cdot 7$ is simple.

(3) Prove that $\mathbb{Q}/\mathbb{Z}$ is not a finitely generated group.

Part 2: Rings

Choose *two* of the following problems.

(4) Let R be a ring. Let I and J be ideals of R, and recall that $I + J = \{a + b \mid a \in I,\, b \in J\}$.

(a) Show that if $I = (S)$ and $J = (T)$ for some subsets $S, T \subseteq R$, then $I + J = (S \cup T)$.

(b) Let $\pi : R \to R/I$ be the quotient homomorphism. Show that $\dfrac{R/I}{\pi(J)} \cong \dfrac{R}{I+J}$.

(5) (a) Prove that a finite integral domain must be a field.

(b) Prove[1] that if R is a commutative ring and $P \subseteq R$ is a prime ideal such that P has finite index as a subgroup of $(R, +)$, then P is a maximal ideal.

(6) Consider the polynomial $f(x) = x^2 + x + [1]_5$ in $\mathbb{Z}/5[x]$. Show that $R = \dfrac{\mathbb{Z}/5[x]}{(f)}$ is a field, and determine the number of elements of R.

[1] Hint: Consider the quotient ring R/P.