DEFINITION: Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d.

- A **system of parameters** for R is a set of d elements $f_1, \dots, f_d \in \mathfrak{m}$ such that $\mathfrak{m} = \sqrt{(f_1, \dots, f_d)}$.
- An element $f \in \mathfrak{m}$ is a **parameter** if it is part of a system of parameters.
- A set of elements is a **partial system of parameters** if it is a subset of some system of parameters.

THEOREM: Let $(R, \mathfrak{m})$ be a Noetherian local ring and $f_1, \dots, f_t \in \mathfrak{m}$. Then

$$\dim(R/(f_1, \dots, f_t)) \geq \dim(R) - t,$$

and equality holds if and only if $f_1, \dots, f_t$ are a partial system of parameters.

(1) Do systems of parameters always exist?

Yes, we showed it last time.

(2) Proof of Theorem:

(a) To prove the inequality, take a system of parameters $\overline{r_1}, \dots, \overline{r_s}$ for $R/(f_1, \dots, f_t)$, and take representatives $r_1, \dots, r_s$ in R. What do you know about s? What can you say about the ideal $(f_1, \dots, f_t, r_1, \dots, r_s)$? Deduce the inequality.

(b) For the ($\Rightarrow$) part of the equality statement, revisit the argument for the inequality.

(c) For the ($\Leftarrow$) part of the equality statement, apply the inequality.

(a) Take a system of parameters $\overline{r_1}, \dots, \overline{r_s}$ for $R/(f_1, \dots, f_t)$, and take representatives $r_1, \dots, r_s$ in R. Note that $s = \dim(R/(f_1, \dots, f_t))$.
Since the only prime containing $(\overline{r_1}, \dots, \overline{r_s})$ in $R/(f_1, \dots, f_t)$ is the maximal ideal, the only prime containing $(f_1, \dots, f_t, r_1, \dots, r_s)$ in R is the maximal ideal, so its radical is $\mathfrak{m}$. Thus, by Krull Height Theorem, $s + t \geq \dim(R)$; i.e., $\dim(R/(f_1, \dots, f_t)) + t \geq \dim(R)$. Rearranging gives the sought inequality.

(b) Suppose that $\dim(R/(f_1, \dots, f_t)) = \dim(R) - t$. Then in the notation of the above, $s + t = \dim(R)$, so $f_1, \dots, f_t, r_1, \dots, r_s$ are a sequence of $\dim(R)$ elements that generate an $\mathfrak{m}$-primary ideal; i.e., they form a system of parameters. So, $f_1, \dots, f_t$ are a partial system of parameters.

(c) Suppose that $f_1, \dots, f_t, r_1, \dots, r_s$ is a system of parameters, so $s + t = \dim(R)$. Then

$$0 = \dim(R/(f_1, \dots, f_t, r_1, \dots, r_s)) \geq \dim(R/(f_1, \dots, f_t)) - s = \dim(R/(f_1, \dots, f_t)) - (\dim(R) - t),$$

so $\dim(R/(f_1, \dots, f_t)) \leq \dim(R) - t$.

(3) The dimension inequality globally:

(a) Let K be a field and $R = \frac{K[X,Y,Z]}{(XY,XZ)}$. Compute $\dim(R)$ and $\dim(R/(x-1))$.

(b) Does localizing the previous example at (x, y, z) give a counterexample to the Theorem?

(c) Let $R = \mathbb{Z}_{(2)}[X]$. Is $\dim(R/(2X - 1)) \geq \dim(R) - 1$?

(a) We have computed $\dim(R) = 2$ before. We have $R/(x-1) \cong \frac{K[X,Y,Z]}{(X-1,XY,XZ)} \cong \frac{K[Y,Z]}{(Y,Z)} \cong K$.

(b) No, since $x-1$ is a unit, so $R/(x-1)$ is zero.

(c) $\dim(R) \geq 2$ on account of $0 \subsetneqq (2) \subsetneqq (2, X)$, but $R/(2X-1) \cong \mathbb{Z}_{(2)}[1/2] \cong \mathbb{Q}$ has dimension 0, so no.

(4) Systems of parameters and "absolutely-min-avoiding sequences": We say that a prime $\mathfrak{p}$ in a Noetherian ring R is **absolutely minimal** if $\dim(R) = \dim(R/\mathfrak{p})$, and write $\mathrm{AMin}(R)$ for the set of absolutely minimal primes. For convenience[1], let us say that $\mathbf{f} = f_1, \dots, f_t$ is an "*absolutely-min-avoiding sequence*" if

$$f_1 \notin \bigcup_{\mathfrak{p} \in \mathrm{AMin}(R)} \mathfrak{p}, \quad \overline{f_2} \notin \bigcup_{\mathfrak{p} \in \mathrm{AMin}(R/(f_1))} \mathfrak{p}, \quad \overline{f_3} \notin \bigcup_{\mathfrak{p} \in \mathrm{AMin}(R/(f_1,f_2))} \mathfrak{p}, \quad \dots \quad , \text{ and } \overline{f_t} \notin \bigcup_{\mathfrak{p} \in \mathrm{AMin}(R/(f_1,\dots,f_{t-1}))} \mathfrak{p}.$$

Prove that $\mathbf{f}$ is a absolutely-min-avoiding sequence if and only if $\mathbf{f}$ is a system of parameters.

This boils down to the observation that $\dim(R/f) < \dim(R)$ if and only if f is not in any absolutely minimal prime.

(5) Systems of parameters vs "height sequences"

(a) Show that a height sequence is a system of parameters.

(b) Let $R = \frac{K[X,Y,Z]_{(x,y,z)}}{(XY,XZ)}$. Show that $y, x+z$ is a system of parameters, but not a height sequence. Now show that $x+z, y$ is a height sequence.

(a) This follows because $\mathrm{height}((f_1, \dots, f_d)) = d$ implies that $\mathrm{Min}((f_1, \dots, f_d)) = \{\mathfrak{m}\}$, so $\sqrt{(f_1, \dots, f_d)} = \mathfrak{m}$.

(b) We have seen earlier that $\sqrt{(y, x+z)} = \mathfrak{m}$. However, y is the minimal prime (y, z), so $\mathrm{height}((y)) = 0$. On the other hand, $x+z$ is not in any minimal prime, so $\mathrm{height}((x+z)) = 1$, and we have already seen $\mathrm{height}((x+z, y)) = 2$.

THEOREM: Let R be a Noetherian ring of finite dimension. Then $\dim(R[X_1, \dots, X_n]) = \dim(R) + n$.

(6) Proof of polynomial theorem:

(a) Explain why it suffices to deal with the case $n = 1$ and $\dim(R) < \infty$.

(b) Explain why $\dim(R[X]) \geq \dim(R) + 1$.

(c) Let $\mathfrak{q} \in \mathrm{Spec}(R[X])$ and $\mathfrak{p} = \mathfrak{q} \cap R$. Explain why the Theorem reduces to the claim that $\mathrm{height}(\mathfrak{q}) \leq \mathrm{height}(\mathfrak{p}) + 1$.

(d) Explain why $\mathfrak{q}R_{\mathfrak{p}}[X]$ is prime and $\mathrm{height}(\mathfrak{q}) = \mathrm{height}(\mathfrak{q}R_{\mathfrak{p}}[X])$.

(e) Explain why the Theorem reduces to

CLAIM: If $(S, \mathfrak{m})$ is a Noetherian *local* ring, and $\mathfrak{a} \in S[X]$ is a prime that contracts to

[1] The term "*absolutely-min-avoiding sequence*" is not real, and has just been made up here to simplify the discussion. However, **absolutely minimal** prime is standard.

$\mathfrak{m}$, then $\dim(S[X]_{\mathfrak{a}}) \leq \dim(S) + 1$.
We retain this setup henceforth.

(f) Let $f_1, \dots, f_d$ be a system of parameters of S. Show[2] that $\dim(\frac{S}{(f_1,\dots,f_d)}[X]) = 1$.

(g) Show that $\dim(S[X]_{\mathfrak{a}}/(f_1, \dots, f_d)) \leq 1$.

(h) Complete the proof.

(a) The general n case follows from the $n = 1$ case by induction. Note that the expansion of a prime in R to $R[X]$ is prime again, so $\dim(R[X]) \geq \dim(R)$, and if $\dim(R)$ is infinite, so is $\dim(R[X])$.

(b) As mentioned above, the expansion of a prime in R to $R[X]$ is prime again, so one can take a chain of primes in R and obtain a chain of the same length in $R[X]$ by expansion. But, an expanded prime $\mathfrak{p}R[X]$ is not maximal since $R[X]/\mathfrak{p}R[X] \cong (R/\mathfrak{p})[X]$ is not a field, so $\dim(R[X]) > \dim(R)$.

(c) If the height of any prime in $R[X]$ is no more than the height of some prime of R, then $\dim(R[X]) \leq \dim(R) + 1$.

(d) For the first, there is a bijection between primes contained in $\mathfrak{p}$ and primes contained in $\mathfrak{p}R_{\mathfrak{p}}$. For the second, $\mathfrak{q}R_{\mathfrak{p}}[X] = (R \smallsetminus \mathfrak{p})^{-1}\mathfrak{q}$ is the localization of a prime, which is prime. For the last, since $\mathfrak{q} \cap R \subseteq \mathfrak{p}$ we have $\mathfrak{q} \cap (R \smallsetminus \mathfrak{p}) = \varnothing$, and likewise for every prime contained in $\mathfrak{q}$. Thus there is a bijection between primes contained in $\mathfrak{q}$ and primes contained in $\mathfrak{q}R_{\mathfrak{p}}[X]$.

(e) Apply the CLAIM with $S = R_{\mathfrak{p}}$ and $\mathfrak{a} = \mathfrak{q}R_{\mathfrak{p}}[X]$. We then have

$$\begin{aligned} \text{height}(\mathfrak{p}) + 1 = \dim(R_{\mathfrak{p}}) + 1 &= \dim(S) + 1 \geq \dim(S[X]_{\mathfrak{a}}) \\ &= \dim(\mathfrak{q}R_{\mathfrak{p}}[X]) = \text{height}(\mathfrak{q}R_{\mathfrak{p}}[X]) = \text{height}(\mathfrak{q}). \end{aligned}$$

(f) Since $\sqrt{(f_1, \dots, f_d)} = \mathfrak{m}$, every element of $\mathfrak{m}$ is nilpotent in $\overline{S} = \frac{S}{(f_1,\dots,f_d)}$. Now, the nilpotents in $\overline{S}[X]$ are the polynomials all of whose coefficients are nilpotent, so the nilradical of $\overline{S}[X]$ is $\mathfrak{m}\overline{S}[X]$. But then

$$\dim(\overline{S}[X]) = \dim(\overline{S}[X]/\mathfrak{m}\overline{S}[X]) = \dim((S/\mathfrak{m})[X]) = 1.$$

(g) $S[X]_{\mathfrak{a}}/(f_1, \dots, f_d)$ is a localization of $S[X]/(f_1, \dots, f_d) \cong \overline{S}[X]$, so the dimension is no larger than 1.

(h) Done!

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(7) Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be Noetherian local rings. Let $\phi : R \to S$ be a homomorphism such that $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$. Prove that $\dim(S) \leq \dim(R) + \dim(S/\phi(\mathfrak{m})S)$.

[2] Hint: Use that $\dim(R) = \dim(R/\sqrt{0})$, and that a polynomial is nilpotent if and only if all of its coefficients are nilpotent. Make sure you understand why both of these are true!