§8.38: SYSTEMS OF PARAMETERS

DEFINITION: Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d.

- A **system of parameters** for R is a set of d elements $f_1, \dots, f_d \in \mathfrak{m}$ such that $\mathfrak{m} = \sqrt{(f_1, \dots, f_d)}$.
- An element $f \in \mathfrak{m}$ is a **parameter** if it is part of a system of parameters.
- A set of elements is a **partial system of parameters** if it is a subset of some system of parameters.

THEOREM: Let $(R, \mathfrak{m})$ be a Noetherian local ring and $f_1, \dots, f_t \in \mathfrak{m}$. Then

$$\dim(R/(f_1, \dots, f_t)) \geq \dim(R) - t,$$

and equality holds if and only if $f_1, \dots, f_t$ are a partial system of parameters.

(1) Do systems of parameters always exist?

(2) Proof of Theorem:
- **(a)** To prove the inequality, take a system of parameters $\overline{r_1}, \dots, \overline{r_s}$ for $R/(f_1, \dots, f_t)$, and take representatives $r_1, \dots, r_s$ in R. What do you know about s? What can you say about the ideal $(f_1, \dots, f_t, r_1, \dots, r_s)$? Deduce the inequality.
- **(b)** For the ($\Rightarrow$) part of the equality statement, revisit the argument for the inequality.
- **(c)** For the ($\Leftarrow$) part of the equality statement, apply the inequality.

(3) The dimension inequality globally:
- **(a)** Let K be a field and $R = \frac{K[X,Y,Z]}{(XY,XZ)}$. Compute $\dim(R)$ and $\dim(R/(x-1))$.
- **(b)** Does localizing the previous example at (x, y, z) give a counterexample to the Theorem?
- **(c)** Let $R = \mathbb{Z}_{(2)}[X]$. Is $\dim(R/(2X-1)) \geq \dim(R) - 1$?

(4) Systems of parameters and "absolutely-min-avoiding sequences": We say that a prime $\mathfrak{p}$ in a Noetherian ring R is **absolutely minimal** if $\dim(R) = \dim(R/\mathfrak{p})$, and write $\mathrm{AMin}(R)$ for the set of absolutely minimal primes. For convenience[1], let us say that $\mathbf{f} = f_1, \dots, f_t$ is an "*absolutely-min-avoiding sequence*" if

$$f_1 \notin \bigcup_{\mathfrak{p} \in \mathrm{AMin}(R)} \mathfrak{p}, \qquad \overline{f_2} \notin \bigcup_{\mathfrak{p} \in \mathrm{AMin}(R/(f_1))} \mathfrak{p}, \qquad \overline{f_3} \notin \bigcup_{\mathfrak{p} \in \mathrm{AMin}(R/(f_1,f_2))} \mathfrak{p}, \quad \dots \quad, \text{ and } \overline{f_t} \notin \bigcup_{\mathfrak{p} \in \mathrm{AMin}(R/(f_1,\dots,f_{t-1}))} \mathfrak{p}.$$

Prove that $\mathbf{f}$ is a absolutely-min-avoiding sequence if and only if $\mathbf{f}$ is a system of parameters.

(5) Systems of parameters vs "height sequences"
- (a) Show that a height sequence is a system of parameters.
- (b) Let $R = \frac{K[X,Y,Z]_{(x,y,z)}}{(XY,XZ)}$. Show that $y, x+z$ is a system of parameters, but not a height sequence. Now show that $x+z, y$ is a height sequence.

---

[1]The term "*absolutely-min-avoiding sequence*" is not real, and has just been made up here to simplify the discussion. However, **absolutely minimal** prime is standard.

THEOREM: Let R be a Noetherian ring of finite dimension. Then $\dim(R[X_1, \dots, X_n]) = \dim(R) + n$.

(6) Proof of polynomial theorem:

- **(a)** Explain why it suffices to deal with the case $n = 1$ and $\dim(R) < \infty$.
- **(b)** Explain why $\dim(R[X]) \geq \dim(R) + 1$.
- **(c)** Let $\mathfrak{q} \in \operatorname{Spec}(R[X])$ and $\mathfrak{p} = \mathfrak{q} \cap R$. Explain why the Theorem reduces to the claim that $\operatorname{height}(\mathfrak{q}) \leq \operatorname{height}(\mathfrak{p}) + 1$.
- **(d)** Explain why $\mathfrak{q}R_{\mathfrak{p}}[X]$ is prime and $\operatorname{height}(\mathfrak{q}) = \operatorname{height}(\mathfrak{q}R_{\mathfrak{p}}[X])$.
- **(e)** Explain why the Theorem reduces to
  CLAIM: If $(S, \mathfrak{m})$ is a Noetherian *local* ring, and $\mathfrak{a} \in S[X]$ is a prime that contracts to $\mathfrak{m}$, then $\dim(S[X]_{\mathfrak{a}}) \leq \dim(S) + 1$.
  We retain this setup henceforth.
- **(f)** Let $f_1, \dots, f_d$ be a system of parameters of S. Show[2] that $\dim(\frac{S}{(f_1,\dots,f_d)}[X]) = 1$.
- **(g)** Show that $\dim(S[X]_{\mathfrak{a}}/(f_1, \dots, f_d)) \leq 1$.
- **(h)** Complete the proof.

(7) Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be Noetherian local rings. Let $\phi : R \to S$ be a homomorphism such that $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$. Prove that $\dim(S) \leq \dim(R) + \dim(S/\phi(\mathfrak{m})S)$.

[2]Hint: Use that $\dim(R) = \dim(R/\sqrt{0})$, and that a polynomial is nilpotent if and only if all of its coefficients are nilpotent. Make sure you understand why both of these are true!