PROPOSITION: Let $R$ be a Noetherian ring, and $\mathfrak{p}$ an ideal of height $h$. Then there exist $f_1, \dots, f_h \in R$ such that $\mathfrak{p}$ is a minimal prime of $(f_1, \dots, f_h)$.

THEOREM: Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then

$$\dim(R) = \min\left\{ t \geq 0 \;\middle|\; \exists f_1, \dots, f_t \in R \text{ such that } \mathfrak{m} = \sqrt{(f_1, \dots, f_t)} \right\}.$$

**(1)** Deduce the Theorem from the Proposition.

The dimension of $R$ is the height of $\mathfrak{m}$. By the Proposition, $\mathfrak{m}$ is a minimal prime of a $d$-generated ideal $I$. But, no other prime $\mathfrak{p}$ can be minimal over $I$, since any other prime satisfies $\mathfrak{p} \subsetneqq \mathfrak{m}$, and $I \subseteq \mathfrak{p}$ contradicts that $\mathfrak{m}$ is a minimal prime. Then $\mathrm{Min}(I) = \{\mathfrak{m}\}$ implies $\sqrt{I} = \mathfrak{m}$.

**(2)** Let $K$ be a field, and $R = \left( \dfrac{K[X,Y,Z]}{(XY, XZ)} \right)_{(x,y,z)}$. Verify that $\dim(R) = 2$ and $\sqrt{(y, x+z)} = (x,y,z)$.

Since $\mathrm{Min}(R) = \{(x), (y,z)\}$ we have

$$\dim(R) = \max\{\dim(R/(x)), \dim(R/(y,z))\} = \max\{2, 1\} = 2.$$

Note that $x^2 = x(x+z)$ and $z^2 = z(x+z)$, so $(x,y,z) \subseteq \sqrt{(y, x+z)}$ and equality must hold.

**(3)** Let $R$ be a Noetherian ring, and $\mathbf{f} = f_1, \dots, f_t \in R$ be a sequence of elements in $R$. For convenience[1], let us say that $\mathbf{f}$ is a "*min-avoiding sequence*" if

$$f_1 \notin \bigcup_{\mathfrak{p} \in \mathrm{Min}((0))} \mathfrak{p}, \qquad f_2 \notin \bigcup_{\mathfrak{p} \in \mathrm{Min}((f_1))} \mathfrak{p}, \qquad f_3 \notin \bigcup_{\mathfrak{p} \in \mathrm{Min}((f_1, f_2))} \mathfrak{p}, \qquad \dots \quad, \text{ and } \; f_t \notin \bigcup_{\mathfrak{p} \in \mathrm{Min}((f_1, \dots, f_{t-1}))} \mathfrak{p};$$

and let us say that $\mathbf{f}$ is a "*height sequence*" if

$$\mathrm{height}((f_1)) = 1, \quad \mathrm{height}((f_1, f_2)) = 2, \quad \dots \quad, \text{ and } \; \mathrm{height}((f_1, f_2, \dots, f_t)) = t.$$

Prove that $\mathbf{f}$ is a min-avoiding sequence if and only if $\mathbf{f}$ is a height sequence.

Suppose that $\mathbf{f}$ is a "min-avoiding sequence". Then $f_1$ is not in any minimal prime of $R$, so every prime containing $f_1$ has height at least one, and by PIT, every minimal prime of $f_1$ at height at most one, so every minimal prime of $f_1$ has height exactly one. Then, proceeding inductively, assume that $(f_1, \dots, f_j)$ has height $j$. By KHT, every minimal prime of $(f_1, \dots, f_j)$ then has height $j$. By assumption $f_{j+1}$ is not in any minimal prime of $(f_1, \dots, f_j)$. Let $\mathfrak{q}$ be a minimal prime of $(f_1, \dots, f_{j+1})$. Then $\mathfrak{q}$ contains $(f_1, \dots, f_j)$, hence contains some minimal prime $\mathfrak{p}$ of $(f_1, \dots, f_j)$, and since

[1]The terms "*min-avoiding sequence*" and "*height sequence*" are not real, and have just been made up here to simplify the discussion.

$f_{j+1} \notin \mathfrak{p}$, we must have $\mathfrak{q} \supsetneq \mathfrak{p}$. Thus $\mathrm{height}(\mathfrak{q}) > \mathrm{height}(\mathfrak{p}) = j$. But by KHT, $\mathrm{height}(\mathfrak{q}) \leq j+1$, so equality most hold. Thus, **f** is a "height sequence".

Now suppose that **f** is a "height sequence". Then $f_1$ is not in any minimal prime of $R$, by definition of height. Suppose for some $j$ that $f_{j+1}$ is some minimal prime $\mathfrak{p}$ of $(f_1, \dots, f_j)$. Since the height of $(f_1, \dots, f_j)$ is $j$, $\mathfrak{p}$ has height at least $j$, but also at most $j$ by KHT, so $\mathrm{height}(\mathfrak{p}) = j$. But $f_{j+1} \in \mathfrak{p}$ implies $(f_1, \dots, f_{j+1}) \subseteq \mathfrak{p}$, and thus $\mathrm{height}((f_1, \dots, f_{j+1})) \leq \mathrm{height}(\mathfrak{p}) = j$, contradicting that we have a hight sequence. We conclude that $f_{j+1}$ is not in any minimal prime of $(f_1, \dots, f_j)$; i.e., that **f** is a "min-avoiding sequence".

**(4)** Let $R$ be a Noetherian ring and $\mathfrak{p}$ a prime of height $h$. Prove that there exists a min-avoiding sequence of $h$ elements in $\mathfrak{p}$, and deduce the Proposition.

If $\mathfrak{p}$ has height $0$, then the empty sequence vacuously works. Otherwise, to construct such a sequence inductively, for $j < h$, we choose $f_{j+1} \in \mathfrak{p}$ but not in any minimal prime of $(f_1, \dots, f_j)$. To see that this is possible, note that $f_1, \dots, f_j \in \mathfrak{p}$ so $(f_1, \dots, f_j) \subseteq \mathfrak{p}$, and the minimal primes of $(f_1, \dots, f_j)$ are primes contained in $\mathfrak{p}$ of height $j < h$, so are properly contained in $\mathfrak{p}$. Since there are finitely many such minimal primes, by prime avoidance, we know that $\mathfrak{p}$ is not contained in the union of these primes. Thus, we can pick $f_{j+1}$ is required.

Now, every minimal prime of $(f_1, \dots, f_h)$ has height $h$, and $(f_1, \dots, f_h) \subseteq \mathfrak{p}$. Thus, there is a minimal prime of $(f_1, \dots, f_h)$ contained in $\mathfrak{p}$ of height $h$, but for height reasons, these must be equal. That is, $\mathfrak{p}$ is a minimal prime of $(f_1, \dots, f_h)$.

(5) Let $R$ be a Noetherian ring of dimension $d$ and $I$ an arbitrary ideal.

(a) Show that if $R$ is local, then there exist $f_1, \dots, f_d \in R$ such that $\sqrt{(f_1, \dots, f_d)} = \sqrt{I}$.

(b) Show that, in general, there exist $f_1, \dots, f_{d+1} \in R$ such that $\sqrt{(f_1, \dots, f_{d+1})} = \sqrt{I}$.