PROPOSITION: Let R be a Noetherian ring, and $\mathfrak{p}$ an ideal of height h. Then there exist $f_1, \dots, f_h \in R$ such that $\mathfrak{p}$ is a minimal prime of $(f_1, \dots, f_h)$.

THEOREM: Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then

$$\dim(R) = \min\left\{ t \geq 0 \,\middle|\, \exists f_1, \dots, f_t \in R \text{ such that } \mathfrak{m} = \sqrt{(f_1, \dots, f_t)} \right\}.$$

(1) Deduce the Theorem from the Proposition.

(2) Let K be a field, and $R = \left(\dfrac{K[X,Y,Z]}{(XY, XZ)}\right)_{(x,y,z)}$. Verify that $\dim(R) = 2$ and $\sqrt{(y, x+z)} = (x, y, z)$.

(3) Let R be a Noetherian ring, and $\mathbf{f} = f_1, \dots, f_t \in R$ be a sequence of elements in R. For convenience[1], let us say that $\mathbf{f}$ is a "*min-avoiding sequence*" if

$$f_1 \notin \bigcup_{\mathfrak{p} \in \operatorname{Min}((0))} \mathfrak{p}, \qquad f_2 \notin \bigcup_{\mathfrak{p} \in \operatorname{Min}((f_1))} \mathfrak{p}, \qquad f_3 \notin \bigcup_{\mathfrak{p} \in \operatorname{Min}((f_1, f_2))} \mathfrak{p}, \qquad \dots \quad , \text{ and } f_t \notin \bigcup_{\mathfrak{p} \in \operatorname{Min}((f_1, \dots, f_{t-1}))} \mathfrak{p};$$

and let us say that $\mathbf{f}$ is a "*height sequence*" if

$$\operatorname{height}((f_1)) = 1, \quad \operatorname{height}((f_1, f_2)) = 2, \quad \dots \quad , \text{ and } \operatorname{height}((f_1, f_2, \dots, f_t)) = t.$$

Prove that $\mathbf{f}$ is a min-avoiding sequence if and only if $\mathbf{f}$ is a height sequence.

(4) Let R be a Noetherian ring and $\mathfrak{p}$ a prime of height h. Prove that there exists a min-avoiding sequence of h elements in $\mathfrak{p}$, and deduce the Proposition.

(5) Let R be a Noetherian ring of dimension d and I an arbitrary ideal.
 (a) Show that if R is local, then there exist $f_1, \dots, f_d \in R$ such that $\sqrt{(f_1, \dots, f_d)} = \sqrt{I}$.
 (b) Show that, in general, there exist $f_1, \dots, f_{d+1} \in R$ such that $\sqrt{(f_1, \dots, f_{d+1})} = \sqrt{I}$.

[1] The terms "*min-avoiding sequence*" and "*height sequence*" are not real, and have just been made up here to simplify the discussion.