§8.36: KRULL HEIGHT THEOREM

PRINCIPAL IDEAL THEOREM: Let R be a Noetherian ring and $f \in R$. Then every minimal prime of (f) has height at most one.

KRULL'S HEIGHT THEOREM: Let R be a Noetherian ring and $I = (f_1, \dots, f_n)$. Then every minimal prime of I has height at most n.

(1) Use Krull's Height Theorem to deduce the following:
- **(a)** Every ideal in a Noetherian ring has finite height.
- **(b)** Every Noetherian local ring has finite dimension.
- **(c)** If R is a finitely generated algebra over a field that is a domain, and $I = (f_1, \dots, f_t)$ is a proper ideal, then $\dim(R/I) \geq \dim(R) - t$.

(2) Proof of Principal Ideal Theorem:
- **(a)** Suppose that the Theorem is false, so there is some Noetherian ring S, some $g \in S$, and some prime $\mathfrak{q}$ such that $\mathfrak{q} \in \mathrm{Min}((g))$ with $\mathrm{height}(\mathfrak{q}) > 1$. Show that we can then find a Noetherian local domain $(R, \mathfrak{m})$ of dimension greater than one and some $f \in R$ such that $\mathrm{Min}((f)) = \{\mathfrak{m}\}$. Henceforth, we continue with this notation.
- **(b)** Explain why $R/(f)$ is Artinian.
- **(c)** Let $\mathfrak{q}$ be a prime between (0) and $\mathfrak{m}$. Let $\mathfrak{q}^{(n)} = \mathfrak{q}^n R_{\mathfrak{q}} \cap R$; recall that, by the Second Uniqueness Theorem for Primary Decomposition, this[1] is the $\mathfrak{q}$-primary component of $\mathfrak{q}^n$ in R in any primary decomposition. Explain why there exists some n such that $\mathfrak{q}^{(n)} R/(f) = \mathfrak{q}^{(n+k)} R/(f)$ for all $k > 0$.
- **(d)** Show[2] that $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+k)} = f(\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+k)})$ for all $k > 0$.
- **(e)** Show that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+k)}$ for all $k > 0$.
- **(f)** Show that $0 \neq \bigcap_{n>0} \mathfrak{q}^{(n)}$ from above and $\bigcap_{n>0} \mathfrak{q}^{(n)} \subseteq \bigcap_{n>0} \mathfrak{q}^n R_{\mathfrak{q}} = 0$ from another Theorem to obtain the decisive contradiction.

(3) Proof of Krull Height Theorem: We induce on t.
- **(a)** Dispatch with the base case.
- **(b)** Fix a chain of primes
$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_h = \mathfrak{p}$$
with $\mathfrak{p}$ a minimal prime $\mathfrak{p}$ of $I = (f_1, \dots, f_t)$. Suppose that $f_1 \in \mathfrak{p}_1$. Apply the inductive hypothesis to $I/(f_1)$ in $R/(f_1)$ and complete the inductive step in this case.
- **(c)** Use the Principal Ideal Theorem to prove the following:
LEMMA: Let R be a Noetherian ring, $\mathfrak{p} \subsetneq \mathfrak{q} \subsetneq \mathfrak{r}$ be primes and $f \in \mathfrak{r}$. Then there exists some prime $\mathfrak{q}'$ such that $\mathfrak{p} \subsetneq \mathfrak{q}' \subsetneq \mathfrak{r}$ and $f \in \mathfrak{q}'$.
- **(d)** Use the Lemma to complete the inductive step in the case $f_1 \notin \mathfrak{p}_1$.

(4) Let K be a field, and $R = K[X, XY, XY^2, \dots] \subseteq S = K[X, Y]$. Show that the height of $(X)R$ is two. Compare this to Krull's Height Theorem.

(5) Let R be a Noetherian ring, I be an ideal, and $f \in R$. Must one have $\mathrm{height}(I + (f)) \leq \mathrm{height}(I) + 1$?

(6) Let R be a Noetherian ring and $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals. Show that if there exists some prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$, then there exist infinitely many primes between $\mathfrak{p}$ and $\mathfrak{q}$.

[1] This is known as the nth *symbolic power* of $\mathfrak{q}$.
[2] Use the fact that $f \notin \mathfrak{q}$ and $\mathfrak{q}^{(n)}$ is primary.