DEFINITION: A ring R is **Artinian** if every descending chain of ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ eventually stabilizes: i.e., there is some N such that $I_n = I_N$ for all $n \geq N$. A module is **Artinian** if every descending chain of submodules $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ eventually stabilizes.

PROPOSITION: Let R be a ring and M be a module.

(1) M is Artinian if and only if every nonempty family $\mathcal{S}$ of submodules of M has a minimal element.
(2) If N is a submodule of M, then M is Artinian if and only if N and M/N are both Artinian.

THEOREM: Let R be a ring. The following are equivalent:

(1) R is Noetherian of dimension zero,
(2) R is a finite product of Noetherian local rings of dimension zero,
(3) R is a finite length R-module,
(4) R is Artinian.

(1) Jordan-Hölder review: Explain why a finite length module is Artinian.

Given such a chain, the length of each successive submodule is smaller, so any such chain can have length at most the length of M.

(2) Proof of the Theorem, the useful part: Prove[1] that (i)⇒(ii)⇒(iii)⇒(iv).

For (i)⇒(ii), if R is Noetherian of dimension zero, then $V(0)$ consists of maximal ideals by the dimension assumption, and is finite since every such ideal is minimal, and Noetherian rings have finitely many minimal primes. Then, by a homework problem, since $V(0)$ is a finite set of maximal ideals, $R = R/0 \cong R/Q_1 \times \cdots \times R/Q_t$ where each Q_t is primary to a maximal ideal, so each R/Q_i is local of dimension zero (and Noetherian, since it is a quotient of a Noetherian ring).

For (ii)⇒(iii), we have that R is a fintiely generated R-module whose support is a finite set of maximal ideals, so R has finite length.

(iii)⇒(iv) follows from the previous problem.

(3) Vector spaces:

(a) Let K be a field and V be a vector space. Show that V is finite-dimensional if and only if V is Noetherian if and only if V is Artinian.
(b) Let $(R, \mathfrak{m}, k)$ be a local ring, and M be an R-module such that $\mathfrak{m}M = 0$. Show that M has finite length if and only if M is Noetherian if and only if M is Artinian.

(a) Finite-dimensional is the same as finite length, and finite length implies Noetherian and Artinian in general. Conversely, in an infinite dimensional vector space, one can take a basis and construct infinite ascending or descending chains of subsets of the basis, and the spans form infinite ascending or descending chains of subspaces, so V is neither Artinian nor Noetherian.

(b) One can identify such a module with a k-module and apply part (1).

[1]Hint: In the setting of (i), note that $V(0)$ is a finite set of maximal ideals, and use a homework problem.

(4) Proof of the Theorem, the fun part: Suppose that R is Artinian.

(a) First, we show $\dim(R) = 0$: By way of contradiction, suppose there is nonmaximal prime $\mathfrak{p}$, so there is some nonzero nonunit $a \in R/\mathfrak{p}$. Consider the descending chain of ideals

$$(a) \supseteq (a^2) \supseteq (a^3) \supseteq \cdots .$$

to obtain a contradiction.

(b) Second, we show that $\operatorname{Max}(R)$ is finite: By way of contradiction, if $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \dots$ are distinct maximal ideals, consider the descending chain of ideals

$$\mathfrak{m}_1 \supseteq \mathfrak{m}_1 \cap \mathfrak{m}_2 \supseteq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \supseteq \cdots .$$

(c) Third, we show that R is a finite product of Artinian local rings of dimension zero: Apply a homework problem.

(d) Fourth, we show that an Artinian local ring $(R, \mathfrak{m}, k)$ has finite length: Consider the chain

$$\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \mathfrak{m}^3 \supseteq \cdots .$$

What do we deduce? Why do we *not* immediate deduce that $\mathfrak{m}^n = 0$ for some n from NAK?

(e) Fourth continued: If $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ and $\mathfrak{m}^n \neq 0$, consider $\mathcal{S} = \{\text{ideals } J \mid J\mathfrak{m}^n \neq 0\}$. Explain why $\mathcal{S}$ has a minimal element I, and I is principal. Now deduce that $\mathfrak{m}^n = 0$.

(f) Fourth continueder: Explain why $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ has finite length for each i. Deduce that R has finite length.

(g) Complete the proof.

(a) Finite-dimensional is the same as finite length, and finite length implies Noetherian and Artinian in general. Conversely, in an infinite dimensional vector space, one can take a basis and construct infinite ascending or descending chains of subsets of the basis, and the spans form infinite ascending or descending chains of subspaces, so V is neither Artinian nor Noetherian.

(b) One can identify such a module with a k-module and apply part (1).

(5) Artinian Modules:

(a) Let K be a field. Show that the $K[X]$-module $K[X]_X/K[X]$ is Artinian but not finite length.

(b) Show that an R-module M has finite length if and only if it is Artinian and Noetherian.

(c) Let R be a Noetherian ring. Show that if M is an Artinian module, then $\operatorname{Ass}_R(M) \subseteq \operatorname{Max}(R)$.

(d) Let R be a Noetherian $\mathbb{N}$-graded ring with $R_0 = K$ a field. Show that if M is an Artinian $\mathbb{Z}$-graded module, then there is some n such that $M_{\geq n} = 0$.

(e) Let R be a Noetherian ring. If M is an Artinian module, must $\operatorname{Ass}_R(M)$ be finite?