DEFINITION: Let $R$ be a ring and $M$ a $R$-module.

(1) $M$ is **simple** if it is nonzero and $M$ has no nontrivial proper submodules.

(2) A **composition series** for $M$ of length $n$ is a chain of submodules

$$M = M_n \supsetneq M_{n-1} \supsetneq \cdots \supsetneq M_1 \supsetneq M_0 = 0$$

with $M_i/M_{i-1}$ simple for all $i = 1, \dots, n$. The

(3) $M$ has **finite length** if it admits a composition series. The **length** of $M$, denoted $\ell_R(M)$ is the minimal length $n$ of a composition series for $M$.

JORDAN-HÖLDER THEOREM: Let $R$ be a ring, and $M$ a module *of finite length*. Let $N \subseteq M$ be a submodule.

(1) Any descending chain of submodules of $M$ can be refined[1] to a composition series for $M$.

(2) Every composition series for $M$ has the same length.

(3) If $N \subseteq M$ is any submodule, then

(a) $N$ and $M/N$ have finite length, and $\ell_R(N), \ell_R(M/N) \leq \ell_R(M)$,

(b) $\ell_R(N), \ell_R(M/N) < \ell_R(M)$ unless $M = N$ or $N = 0$ respectively, and

(c) $\ell_R(N) + \ell_R(M/N) = \ell_R(M)$.

COROLLARY: If $M$ has finite length, then $M$ is Noetherian and any descending chain of submodules of $M$ stabilizes.

LEMMA: Let $R$ be a ring. A module $M$ is simple if and only if $M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

PROPOSITION: Let $R$ be a Noetherian ring, and $M$ be a module. The following are equivalent:

(1) $M$ has finite length,

(2) $M$ is finitely generated and $\mathrm{Supp}_R(M) \subseteq \mathrm{Max}(R)$,

(3) $M$ is finitely generated and $\mathrm{Ass}_R(M) \subseteq \mathrm{Max}(R)$.

**(1)** Working with length: Let $R = \mathbb{R}[X, Y]$.

**(a)** Compute a composition series and find the $R$-module length of $M = R/(X^2 + 1, Y)$.

**(b)** Compute a composition series and find the $R$-module length of $M = R/(X^2 + X, Y)$.

**(c)** Compute a composition series and find the $R$-module length of $M = (X, Y)/(X^2, Y^2)$.

**(a)** $(X^2 + 1, Y)$ is a maximal ideal, so $0 \subsetneq M$ is a composition series and $M$ has length one (is simple).

**(b)** We can take $0 \subsetneq (X + 1, Y)/(X^2 + X, Y) \subsetneq M$. The quotients are isomorphic to $R/(X, Y)$ and $R/(X + 1, Y)$, respectively, so this is a composition series. The length is two.

**(c)** We can take $0 \subsetneq (X^2, XY, Y^2)/(X^2, Y^2) \subsetneq (X, Y^2)/(X^2, Y^2) \subsetneq M$. Each quotient is isomorphic to $R/(X, Y)$. The length is three.

**(2)** Use the Jordan-Hölder Theorem to prove the Corollary.

[1] That is, terms can be inserted in between others in the chain to get a composition series.

Given an ascending chain, the lengths of the successive modules increase, so any such chain can have length at most the length of $M$. Given such a chain, the length of each successive submodule is smaller, so any such chain can have length at most the length of $M$.

**(3)** Proof of Proposition: Let $R$ be a Noetherian ring.
   **(a)** How do the concepts of "composition series" and "prime filtration" compare?
   **(b)** Why does having finite length imply that $M$ is finitely generated[2]? What can one deduce about the associated primes of $M$? Deduce (1)⇒(3).
   **(c)** Use the definition of support to explain why, if $R/\mathfrak{p}$ is a factor in a prime filtration for $M$, then $\mathfrak{p} \in \mathrm{Supp}_R(M)$. Deduce (2)⇒(1).
   **(d)** Show (3)⇒(2) to complete the proof.

**(a)** A composition series is a special prime filtration.
**(b)** From above, finite length implies Noetherian, and hence finite generation. By assumption, $M$ has a prime filtration with all maximal factors. Since the associated primes are contained in the factors of a prime filtration, $\mathrm{Ass}_R(M) \subseteq \mathrm{Max}(R)$.
**(c)** Given a prime filtration for a module, if we localize at any prime factor $\mathfrak{p}$, then we get a chain of submodules of $M_\mathfrak{p}$, and since $(R/\mathfrak{p})_\mathfrak{p} \neq 0$, some containment is proper in the chain, so $M_\mathfrak{p} \neq 0$. Thus, if $\mathrm{Supp}_R(M) \subseteq \mathrm{Max}(R)$ and $M$ is finitely generated, $M$ has a prime filtration, and any prime filtration for $M$ has only maximal factors.
**(d)** This follows since every prime in the support contains an associated prime.

(4) Show that if $R$ is a finitely generated algebra of an algebraically closed field $K$, then the length of an $R$-module $M$ is equal to the dimension of $M$ as a $K$-vector space.

(5) Proof of Jordan-Hölder: We will show (3a), (3b) directly, then deduce (1), (2), and (3c).
   (a) Let's start with deducing the other parts from (3a) and (3b). Show that (3a)+(3b)⇒(1) by inducing on length.
   (b) Show that (3a)⇒(2) by induction on length: given another composition series
$$M = N_m \supsetneqq N_{m-1} \supsetneqq \cdots \supsetneqq N_1 \supsetneqq N_0 = 0,$$
   consider the case $N_{m-1} = M_{n-1}$, and in the other case, consider $K = N_{m-1} \cap M_{n-1}$.
   (c) Show that (1)+(2)⇒(3c).
   (d) Now we start on (3a) and (3b). Use the Second Isomorphism Theorem to show that
$$\frac{M_i \cap N}{M_{i-1} \cap N} \cong \frac{M_i \cap N + M_{i-1}}{M_{i-1}}.$$
   (e) Show that $N$ has a composition series of length at most $n$.
   (f) Show that if the composition series you just found for $N$ has length $n$, then $N = M$, so if $N \subsetneqq M$, then $\ell_R(N) < \ell_R(M)$.
   (g) Use the Second Isomorphism Theorem to show that
$$\frac{(M_i + N)/N}{(M_{i-1} + N)/N} \cong \frac{M_i}{M_i \cap (M_{i-1} \cap N)}.$$
   (h) Show that $M/N$ has a composition series of length at most $n$.

[2]The Corollary is fair game.

(i) Show that if the composition series you just found for $M/N$ has length $n$, then $N = 0$, so if $N \neq 0$, then $\ell_R(M/N) < \ell_R(M)$. Deduce (3a) and (3b) to finish the proof.

(a) If $M$ has length one, then $M$ is simple, so any chain of submodules is already a composition series. In general, given a proper chain of submodules $0 = L_0 \subsetneqq \cdots \subsetneqq L_t = M$, we have $\ell(L_i/L_{i-1}) < \ell(M)$ by using (3a) and (3b). By induction on length, we can find composition series for $L_i/L_{i-1}$. Then, by the lattice isomorphism theorem, we can pull back to get chains of submodules from $L_{i-1}$ to $L_i$ with simple quotients. This gives the sought refinement.

(b) If $M$ has length one, again this is trivial. Given another composition series given another composition series

$$M = N_m \supsetneqq N_{m-1} \supsetneqq \cdots \supsetneqq N_1 \supsetneqq N_0 = 0,$$

first consider the case $N_{m-1} = M_{n-1} =: K$. Then $\ell(K) < \ell(M)$, so by induction on length, we can assume that any two composition series for $K$ have the same length; in particular, chain of $N_i$ up to $N_{m-1}$ and the chain of $M_i$ up to $M_{n-1}$ have the same length, so $m = n$.

Now suppose that $N_{m-1} \neq M_{n-1}$, and set $K := N_{m-1} \cap M_{n-1}$. By the second isomorphism theorem, we then have

$$\frac{M}{M_{n-1}} = \frac{M_{n-1} + N_{m-1}}{M_{n-1}} \cong \frac{N_{m-1}}{K}$$

and similarly $M/N_{m-1} \cong M_{n-1}/K$, and both of these modules are simple. Given a composition series for $K$ of length $t$, one obtains a composition series for $M_{n-1}$ of length $t+1$ and a composition series for $N_{m-1}$ of length $t+1$. Since $\ell(M_{n-1}), \ell(N_{m-1}) < \ell(M)$, by induction on length we can assume that $n - 1 = t + 1 = m - 1$ and we conclude that $m = n$.

(c) Refine the chain $0 \subseteq N \subseteq M$ to a composition series of $M$. The portion from $0$ up to $N$ is a composition series for $N$ and the part from $N$ to $M$ yields, in the quotient, a composition series of $M/N$. Since the lengths of any composition series of the same module are the same, the result follows.

(d)

$$\frac{M_i \cap N}{M_{i-1} \cap N} = \frac{M_i \cap N}{(M_i \cap N) \cap M_{i-1}} \cong \frac{M_i \cap N + M_{i-1}}{M_{i-1}}.$$

(e) By the previous part, $\frac{M_i \cap N}{M_{i-1} \cap N}$ is isomorphic to a submodule of $M_i/M_{i-1}$, so it is either simple or zero. It follows that, after removing redundant terms,

$$0 = M_0 \cap N \subseteq M_1 \cap N \subseteq \cdots \subseteq M_n \cap N = N$$

is a composition series for $N$.

(f) If no term is redundant in the chain above, then $\frac{M_i \cap N}{M_{i-1} \cap N} \cong M_i/M_{i-1}$ for all $i$, and arguing inductively on $i$, one has $M_i = M_i \cap N$ for all $i$, so $M = N$.

(g)

$$\frac{(M_i + N)/N}{(M_{i-1} + N)/N} \cong \frac{M_i + N}{M_{i-1} + N} \cong \frac{M_i + (M_{i-1} + N)}{M_{i-1} + N} \cong \frac{M_i}{M_i \cap (M_{i-1} + N)}.$$

(h) From the above, each module $\frac{(M_i+N)/N}{(M_{i-1}+N)/N}$ is isomorphic to a quotient of $M_i/M_{i-1}$, so is either simple of zero. Thus, after removing redundant terms,

$$0 = (M_0 + N)/N \subseteq (M_1 + N)/N \subseteq \cdots \subseteq (M_n + N)/N = M/N$$

is a composition series for $M/N$.

(i) If no term above is redundant, then $M_i \cap (M_{i-1} + N) = M_{i-1}$ for all $i$, so by descending induction on $i$, $N \subseteq M_{i-1}$ for each $i$, and $N = 0$.