DEFINITION: Let R be a ring and M a R-module.

(1) M is **simple** if it is nonzero and M has no nontrivial proper submodules.

(2) A **composition series** for M of length n is a chain of submodules

$$M = M_n \supsetneq M_{n-1} \supsetneq \cdots \supsetneq M_1 \supsetneq M_0 = 0$$

with M_i/M_{i-1} simple for all $i = 1, \dots, n$. The

(3) M has **finite length** if it admits a composition series. The **length** of M, denoted $\ell_R(M)$ is the minimal length n of a composition series for M.

JORDAN-HÖLDER THEOREM: Let R be a ring, and M a module *of finite length*. Let $N \subseteq M$ be a submodule.

(1) Any descending chain of submodules of M can be refined[1] to a composition series for M.

(2) Every composition series for M has the same length.

(3) If $N \subseteq M$ is any submodule, then
 (a) N and M/N have finite length, and $\ell_R(N), \ell_R(M/N) \leq \ell_R(M)$,
 (b) $\ell_R(N), \ell_R(M/N) < \ell_R(M)$ unless $M = N$ or $N = 0$ respectively, and
 (c) $\ell_R(N) + \ell_R(M/N) = \ell_R(M)$.

COROLLARY: If M has finite length, then M is Noetherian and any descending chain of submodules of M stabilizes.

LEMMA: Let R be a ring. A module M is simple if and only if $M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

PROPOSITION: Let R be a Noetherian ring, and M be a module. The following are equivalent:

(1) M has finite length,

(2) M is finitely generated and $\operatorname{Supp}_R(M) \subseteq \operatorname{Max}(R)$,

(3) M is finitely generated and $\operatorname{Ass}_R(M) \subseteq \operatorname{Max}(R)$.

(1) Working with length: Let $R = \mathbb{R}[X, Y]$.
 (a) Compute a composition series and find the R-module length of $M = R/(X^2 + 1, Y)$.
 (b) Compute a composition series and find the R-module length of $M = R/(X^2 + X, Y)$.
 (c) Compute a composition series and find the R-module length of $M = (X, Y)/(X^2, Y^2)$.

(2) Use the Jordan-Hölder Theorem to prove the Corollary.

(3) Proof of Proposition: Let R be a Noetherian ring.
 (a) How do the concepts of "composition series" and "prime filtration" compare?
 (b) Why does having finite length imply that M is finitely generated[2]? What can one deduce about the associated primes of M? Deduce (1)⇒(3).
 (c) Use the definition of support to explain why, if $R/\mathfrak{p}$ is a factor in a prime filtration for M, then $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Deduce (2)⇒(1).
 (d) Show (3)⇒(2) to complete the proof.

[1] That is, terms can be inserted in between others in the chain to get a composition series.

[2] The Corollary is fair game.

(4) Show that if R is a finitely generated algebra of an algebraically closed field K, then the length of an R-module M is equal to the dimension of M as a K-vector space.

(5) Proof of Jordan-Hölder: We will show (3a), (3b) directly, then deduce (1), (2), and (3c).
  (a) Let's start with deducing the other parts from (3a) and (3b). Show that (3a)+(3b)⇒(1) by inducing on length.
  (b) Show that (3a)⇒(2) by induction on length: given another composition series
$$M = N_m \supsetneq N_{m-1} \supsetneq \cdots \supsetneq N_1 \supsetneq N_0 = 0,$$
consider the case $N_{m-1} = M_{n-1}$, and in the other case, consider $K = N_{m-1} \cap M_{n-1}$.
  (c) Show that (1)+(2)⇒(3c).
  (d) Now we start on (3a) and (3b). Use the Second Isomorphism Theorem to show that
$$\frac{M_i \cap N}{M_{i-1} \cap N} \cong \frac{M_i \cap N + M_{i-1}}{M_{i-1}}.$$
  (e) Show that N has a composition series of length at most n.
  (f) Show that if the composition series you just found for N has length n, then $N = M$, so if $N \subsetneq M$, then $\ell_R(N) < \ell_R(M)$.
  (g) Use the Second Isomorphism Theorem to show that
$$\frac{(M_i + N)/N}{(M_{i-1} + N)/N} \cong \frac{M_i}{M_i \cap (M_{i-1} \cap N)}.$$
  (h) Show that M/N has a composition series of length at most n.
  (i) Show that if the composition series you just found for M/N has length n, then $N = 0$, so if $N \neq 0$, then $\ell_R(M/N) < \ell_R(M)$. Deduce (3a) and (3b) to finish the proof.