DEFINITION: Let $K \subseteq L$ be an extension of fields and let S be a subset of L.

(1) The **subfield of L generated by K and S**, denoted $K(S)$, is the smallest subfield of L containing K and S. Equivalently, $K(S)$ is the set of elements in L that can be written as rational function expressions in S with coefficients in K.
(2) We say that S is **algebraically independent** over K if there are nonzero polynomial relations on any finite subset of S. Equivalently, S is algebraically independent over K if, for a set of indeterminates $X = \{X_s \mid s \in S\}$, there is an isomorphism of field extensions of K between the field of rational functions $K(S)$ and $K(X)$ via $s \mapsto X_s$.
(3) We say that S is a **transcendence basis** for L over K if S is algebraically independent over K and the field extension $K(S) \subseteq L$ is algebraic.

LEMMA: Let $K \subseteq L$ be an extension of fields.

(1) Every K-algebraically independent subset of L is contained in a transcendence basis. In particular, there exists a transcendence basis for L over K.
(2) Every transcendence basis for L over K has the same cardinality.

DEFINITION: Let $K \subseteq L$ be an extension of fields. The **transcendence degree** of L over K is the cardinality of a transcendence basis for L over K.

THEOREM: Let K be a field, and R be a domain that is algebra-finite over K. Then, the dimension of R is equal to the transcendence degree of $\mathrm{Frac}(R)$ over K.

(1) Let K be a field, and R be a domain that is algebra-finite over K.
- **(a)** Explain why, if $R = K[f_1, \dots, f_m]$, then $\mathrm{Frac}(R) = K(f_1, \dots, f_m)$.
- **(b)** Show[1] that if $A = K[z_1, \dots, z_t]$ is a Noether normalization for R, then $\{z_1, \dots, z_t\}$ forms a transcendence basis for $\mathrm{Frac}(R)$.
- **(c)** Deduce the Theorem.

(a) Since $f_1, \dots, f_m \in \mathrm{Frac}(R)$, the containment $\mathrm{Frac}(R) \supseteq K(f_1, \dots, f_m)$ holds. Conversely, every element of $\mathrm{Frac}(R)$ can be written as a fraction of elements of R, and an element of R can be written as a polynomial expression in f_i, so each element of $\mathrm{Frac}(R)$ is a rational expression in the f_i's.

(b) By definition, the z_i are algebraically independent. Write $R = \sum A r_i$. We claim that $\mathrm{Frac}(R) = \sum \mathrm{Frac}(A) r_i$. Indeed, given r/s for $r, s \in R$, we can write $st = a$ for some $a \in A$ nonzero and $t \in R$. Then for

[1] Hint: Recall that every nonzero $r \in R$ has a nonzero multiple in A.

some $s_i \in R$, we have $r/s = rt/a = (\sum r_i s_i)/a = \sum (s_i/a) r_i$, so $r/s \in \sum \mathrm{Frac}(A) r_i$.

(c) Follows from the Theorem that in this setting the dimension equals the cardinality of the variables in a Noether normalization, and that the transcendence degree of the fraction field of a NN is the number of elements in the NN.

(2) Let K be a field. Use the Theorem to compute the dimension of

$$R = K[UX, UY, UZ, VX, VY, VZ] \subseteq K[U, V, X, Y, Z].$$

We have $\mathrm{Frac}(R) = K(UX, UY, UZ, VX, VY, VZ) = K(UX, Y/X, Z/X, V/U)$, which has transcendence degree four.

(3) Let $R \subseteq S$ be domains.

(a) Use the Theorem to prove that if $R \subseteq S$ are finitely generated algebras over some field K, then $\dim(R) \leq \dim(S)$.

(b) Give an example where $\dim(R) > \dim(S)$.

(a) This follows from the transcendence degree characterization, since a maximal algebraically independent subset of $\mathrm{Frac}(R)$ is contained in a maximal algebraically independent subset of $\mathrm{Frac}(S)$.

(b) $\mathbb{Z} \subseteq \mathbb{Q}$.

(4) Proof of Lemma: Let $K \subseteq L$ be fields, and S a subset of L.

(a) Show that S is a transcendence basis for L over K if and only if it is a maximal K-algebraically independent subset of L.

(b) Deduce part (1) of the Lemma.

(c) Show that, to prove part (2) (in the case of two finite transcendence bases), it suffices to show the following
EXCHANGE LEMMA: If $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ are two transcendence bases, then there is some j such that
$\{x_j, y_2, \dots, y_n\}$ is a transcendence basis.

(d) In the setting of the Exchange Lemma, explain why for each j, there is some nonzero $p_j(t) \in K[y_1, \dots, y_n][t]$ such that $p_j(x_j) = 0$.

(e) In the setting of the previous part, explain why there is some j such that $p_j(t) \notin K[y_2, \dots, y_n][t]$.

(f) Show that the conclusion of the Exchange Lemma holds for j as in the previous part.

(a) If $\{l_\lambda\}$ and $l \in L$, then l is algebraic over $K(\{l_\lambda\})$, so there is a nonzero polynomial relation $l^n + r_1 l^{n-1} + \cdots + r_n = 0$ with $r_i \in K(\{l_\lambda\})$. Writing $r_i = \frac{p_i}{q_i}$ and multiplying by the product of the q_i's gives a nonzero polynomial relation on the l_λ's and l. Thus, $\{l_\lambda\}$ is a maximal algebraic subset. The converse is similar.

(b) Given a nested union of algebraically independent subsets, the union is as well, since a relation on one of these sets involves finitely many elements, all of which must occur in one of the sets in the chain. The claim then follows from Zorn's Lemma.

(c) If $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ are two transcendence bases, say that $m \leq n$. If the intersection has $s < m$ elements, then without loss of generality $y_1 \notin \{x_1, \dots, x_m\}$. Then, for some i, $\{x_i, y_2 \dots, y_n\}$ is a transcendence basis, and $\{x_1, \dots, x_m\} \cap \{x_i, y_2 \dots, y_n\}$ has $s+1$ elements. Replacing $\{y_1, \dots, y_n\}$ with $\{x_i, y_2 \dots, y_n\}$ and repeating this process, we obtain a transcendence basis with n elements such that $\{x_1, \dots, x_m\} \subseteq \{y_1, \dots, y_n\}$. But we must then have that these two transcendence bases are equal, so $m = n$.

(d) Since L is algebraic over $K(y_1, \dots, y_n)$, for each i there is some $p_i(t) \in K(y_1, \dots, y_n)[t]$ such that $p_i(x_i) = 0$. We can clear denominators to assume without loss of generality that $p_i(x_i) \in K[y_1, \dots, y_n][t]$.

(e) If not, so $p_i(t) \in K[y_2, \dots, y_n][t]$ for all i, note that each x_i is algebraic over $K(y_2, \dots, y_n)$. Thus, $K(x_1, \dots, x_m)$ is algebraic over $K(y_2, \dots, y_n)$, and since L is algebraic over $K(x_1, \dots, x_m)$, y is algebraic over $K(y_2, \dots, y_n)$, which contradicts that $\{y_1, \dots, y_n\}$ is a transcendence basis. This shows the claim.

(f) Thinking of the equation $p_i(x_i) = 0$ as a polynomial expression in $K[x_i, y_2, \ldots, y_n][y_1]$, y_1 is algebraic over $K(x_i, y_2, \ldots, y_n)$, hence $K(y_1, \ldots, y_n)$ is algebraic over $K(x_i, y_2, \ldots, y_n)$, and L as well.

If $\{x_i, y_2, \ldots, y_n\}$ were algebraically dependent, take a polynomial equation $p(x_i, y_2, \ldots, y_n) = 0$. Note that this equation must involve x_i, since $y_2, \ldots, y_n$ are algebraically independent. We would then have $K(x_i, y_2, \ldots, y_n)$ is algebraic over $K(y_2, \ldots, y_n)$. But since y_1 is algebraic over $K(x_i, y_2, \ldots, y_n)$, we would have that $K(y_1, \ldots, y_n)$ is algebraic over $K(y_2, \ldots, y_n)$, which would contradict that $y_1, \ldots, y_n$ is a transcendence basis.