DEFINITION: Let $K \subseteq L$ be an extension of fields and let S be a subset of L.

(1) The **subfield of L generated by K and S**, denoted $K(S)$, is the smallest subfield of L containing K and S. Equivalently, $K(S)$ is the set of elements in L that can be written as rational function expressions in S with coefficients in K.
(2) We say that S is **algebraically independent** over K if there are nonzero polynomial relations on any finite subset of S. Equivalently, S is algebraically independent over K if, for a set of indeterminates $X = \{X_s \mid s \in S\}$, there is an isomorphism of field extensions of K between the field of rational functions $K(S)$ and $K(X)$ via $s \mapsto X_s$.
(3) We say that S is a **transcendence basis** for L over K if S is algebraically independent over K and the field extension $K(S) \subseteq L$ is algebraic.

LEMMA: Let $K \subseteq L$ be an extension of fields.

(1) Every K-algebraically independent subset of L is contained in a transcendence basis. In particular, there exists a transcendence basis for L over K.
(2) Every transcendence basis for L over K has the same cardinality.

DEFINITION: Let $K \subseteq L$ be an extension of fields. The **transcendence degree** of L over K is the cardinality of a transcendence basis for L over K.

THEOREM: Let K be a field, and R be a domain that is algebra-finite over K. Then, the dimension of R is equal to the transcendence degree of $\mathrm{Frac}(R)$ over K.

(1) Let K be a field, and R be a domain that is algebra-finite over K.
- **(a)** Explain why, if $R = K[f_1, \dots, f_m]$, then $\mathrm{Frac}(R) = K(f_1, \dots, f_m)$.
- **(b)** Show[1] that if $A = K[z_1, \dots, z_t]$ is a Noether normalization for R, then $\{z_1, \dots, z_t\}$ forms a transcendence basis for $\mathrm{Frac}(R)$.
- **(c)** Deduce the Theorem.

(2) Let K be a field. Use the Theorem to compute the dimension of

$$R = K[UX, UY, UZ, VX, VY, VZ] \subseteq K[U, V, X, Y, Z].$$

(3) Let $R \subseteq S$ be domains.
- **(a)** Use the Theorem to prove that if $R \subseteq S$ are finitely generated algebras over some field K, then $\dim(R) \leq \dim(S)$.
- **(b)** Give an example where $\dim(R) > \dim(S)$.

[1]Hint: Recall that every nonzero $r \in R$ has a nonzero multiple in A.

(4) Proof of Lemma: Let $K \subseteq L$ be fields, and S a subset of L.

(a) Show that S is a transcendence basis for L over K if and only if it is a maximal K-algebraically independent subset of L.

(b) Deduce part (1) of the Lemma.

(c) Show that, to prove part (2) (in the case of two finite transcendence bases), it suffices to show the following
EXCHANGE LEMMA: If $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ are two transcendence bases, then there is some j such that
$\{x_j, y_2, \ldots, y_n\}$ is a transcendence basis.

(d) In the setting of the Exchange Lemma, explain why for each j, there is some nonzero $p_j(t) \in K[y_1, \ldots, y_n][t]$ such that $p_j(x_j) = 0$.

(e) In the setting of the previous part, explain why there is some j such that
$p_j(t) \notin K[y_2, \ldots, y_n][t]$.

(f) Show that the conclusion of the Exchange Lemma holds for j as in the previous part.