THEOREM: Let K be a field, and R be a domain that is algebra-finite over K. Let $K[f_1, \dots, f_n]$ be a Noether normalization of R. Any saturated chain of primes from 0 to a maximal ideal $\mathfrak{m}$ of R has length n.

COROLLARY: Let K be a field, and R be a finitely generated K-algebra. Then

(1) For any primes $\mathfrak{p} \subseteq \mathfrak{q}$ of R, every saturated chain of primes from $\mathfrak{p}$ to $\mathfrak{q}$ has the same length. (That is, R is **catenary**).

(2) If R is a domain, and I is an arbitrary ideal, then $\dim(R) = \dim(R/I) + \operatorname{height}(I)$.

(1) Consequences of the Theorem: Let K be a field.

- **(a)** Use the Theorem to deduce that $\dim(K[X_1, \dots, X_n]) = n$.
- **(b)** Use the Theorem to deduce that every Noether normalization has the same number of elements.
- **(c)** Use part (a) above to show that the dimension of a K-algebra is at most the number of generators in an K-algebra generating set.
- **(d)** Use the Theorem to prove part (1) of the Corollary.

- **(a)** Note that $K[X_1, \dots, X_n]$ is a Noether normalization of itself. So, any saturated chain of primes from 0 to a maximal ideal has length n.
- **(b)** Follows because every Noether normalization has cardinality equal to the length of a saturated chain from 0 to any maximal ideal.
- **(c)** If R is a K algebra with n generators, it is a quotient of $K[X_1, \dots, X_n]$, which has dimension n, so R has dimension at most n.
- **(d)** Take two saturated chains of primes from $\mathfrak{p}$ to $\mathfrak{q}$. Let $S = R/\mathfrak{p}$, which is a domain. We get two saturated chains of primes from 0 to $\mathfrak{q}/\mathfrak{p}$ in S. Fix a maximal ideal of S containing $\mathfrak{q}/\mathfrak{p}$. By concatenation, we get two saturated chains from 0 to fixed maximal ideals in S, which must have the same length. So the chains from 0 to $\mathfrak{q}/\mathfrak{p}$ have the same length, and hence the chains from $\mathfrak{p}$ to $\mathfrak{q}$ have the same length.

(2) Let K be a field. Use the Theorem and previous computations to compute the dimension of each of the following rings:

- **(a)** $\dfrac{K[X,Y,Z]}{(X^3+Y^3+Z^3)}$.
- **(b)** $\dfrac{K[X,Y]}{(XY)}$.
- **(c)** $K[X^4, X^3Y, XY^3, Y^4]$.

- **(a)** A Noether normalization is $K[x,y]$, so the dimension is 2.
- **(b)** A Noether normalization is $K[x+y]$, so the dimension is 1.
- **(c)** A Noether normalization is $K[X^4, Y^4]$, so the dimension is 2.

(3) Proof of Theorem: Induce on the number of elements n in a Noether normalization.

(a) Explain the case $n = 0$.

(b) For the general case, let $A = K[z_1, \dots, z_n] \subseteq R$ be a Noether normalization, and take a saturated chain of primes of R:
$$(0) = \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_s = \mathfrak{m}.$$
Explain why $\mathfrak{p}_1$ has height 1.

(c) Explain why $\mathfrak{p}_1 \cap A$ has height 1.

(d) Explain why $\mathfrak{p}_1 \cap A$ is principal.

(e) Explain why, after a change of coordinates, we can assume that $K[z_1, \dots, z_{n-1}]$ is a Noether normalization of $R/\mathfrak{p}_1$.

(f) Finish the proof.

> **(a)** If $n = 0$, then R is a domain module-finite over a field, so a field. Then there are no chains of primes.
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> **(b)** This follows from the definition of saturated.
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> **(c)** This is a Corollary of Going Down.
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> **(d)** From the homework, every height one prime in a UFD is principal.
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> **(e)** Let $\mathfrak{p}_1 \cap A = (f)$. After a change of coordinates in the z_i's, we can assume that f is monic in z_n. We have $K[z_1, \dots, z_n]/(f) \hookrightarrow R/\mathfrak{p}_1$, and this is integral since $A \to R$ is. Then $K[z_1, \dots, z_{n-1}] \hookrightarrow K[z_1, \dots, z_n]/(f)$ is module-finite, and then the composition is a Noether normalization.
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> **(f)** By the induction hypothesis, any saturated chain from $\mathfrak{p}_1/\mathfrak{p}_1$ to a maximal ideal of $R/\mathfrak{p}_1$ has length $n - 1$. So $s = n$.

(4) Use the Theorem to prove part (2) of the Corollary.

(5) Let $R = K[X_1, \dots, X_n]$ and $f_{m+1}, \dots, f_n$ be polynomials such that $f_{m+1} \in K[X_1, \dots, X_{m+1}]$ is monic in X_{m+1}, ..., $f_n \in K[X_1, \dots, X_n]$ is monic in X_n. Show that $K[x_1, \dots, x_m]$ is a Noether normalization for $S = R/(f_{m+1}, \dots, f_n)$, and deduce that $\dim(S) = m$, and that $\text{height}(f_{m+1}, \dots, f_n) = n - m$.

(6) Let K be a field, and let $R \subseteq S$ be an inclusion of finitely generated K-algebras that are both domains. Show that for any $\mathfrak{q} \in \text{Spec}(S)$, $\text{height}(\mathfrak{q}) = \text{height}(\mathfrak{q} \cap R)$.

(7) Let K be a field. Show that $K[\![X_1, \dots, X_n]\!]$ is a domain of dimension n.