THEOREM: Let K be a field, and R be a domain that is algebra-finite over K. Let $K[f_1, \dots, f_n]$ be a Noether normalization of R. Any saturated chain of primes from 0 to a maximal ideal $\mathfrak{m}$ of R has length n.

COROLLARY: Let K be a field, and R be a finitely generated K-algebra. Then

(1) For any primes $\mathfrak{p} \subseteq \mathfrak{q}$ of R, every saturated chain of primes from $\mathfrak{p}$ to $\mathfrak{q}$ has the same length. (That is, R is **catenary**).

(2) If R is a domain, and I is an arbitrary ideal, then $\dim(R) = \dim(R/I) + \mathrm{height}(I)$.

(1) Consequences of the Theorem: Let K be a field.
- **(a)** Use the Theorem to deduce that $\dim(K[X_1, \dots, X_n]) = n$.
- **(b)** Use the Theorem to deduce that every Noether normalization has the same number of elements.
- **(c)** Use part (a) above to show that the dimension of a K-algebra is at most the number of generators in an K-algebra generating set.
- **(d)** Use the Theorem to prove part (1) of the Corollary.

(2) Let K be a field. Use the Theorem and previous computations to compute the dimension of each of the following rings:
- **(a)** $\dfrac{K[X, Y, Z]}{(X^3 + Y^3 + Z^3)}$.
- **(b)** $\dfrac{K[X, Y]}{(XY)}$.
- **(c)** $K[X^4, X^3Y, XY^3, Y^4]$.

(3) Proof of Theorem: Induce on the number of elements n in a Noether normalization.
- **(a)** Explain the case $n = 0$.
- **(b)** For the general case, let $A = K[z_1, \dots, z_n] \subseteq R$ be a Noether normalization, and take a saturated chain of primes of R:

$$(0) = \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_s = \mathfrak{m}.$$

  Explain why $\mathfrak{p}_1$ has height 1.
- **(c)** Explain why $\mathfrak{p}_1 \cap A$ has height 1.
- **(d)** Explain why $\mathfrak{p}_1 \cap A$ is principal.
- **(e)** Explain why, after a change of coordinates, we can assume that $K[z_1, \dots, z_{n-1}]$ is a Noether normalization of $R/\mathfrak{p}_1$.
- **(f)** Finish the proof.

(4) Use the Theorem to prove part (2) of the Corollary.

(5) Let $R = K[X_1, \dots, X_n]$ and $f_{m+1}, \dots, f_n$ be polynomials such that $f_{m+1} \in K[X_1, \dots, X_{m+1}]$ is monic in X_{m+1}, …, $f_n \in K[X_1, \dots, X_n]$ is monic in X_n. Show that $K[x_1, \dots, x_m]$ is a Noether normalization for $S = R/(f_{m+1}, \dots, f_n)$, and deduce that $\dim(S) = m$, and that $\mathrm{height}(f_{m+1}, \dots, f_n) = n - m$.

(6) Let K be a field, and let $R \subseteq S$ be an inclusion of finitely generated K-algebras that are both domains. Show that for any $\mathfrak{q} \in \operatorname{Spec}(S)$, $\operatorname{height}(\mathfrak{q}) = \operatorname{height}(\mathfrak{q} \cap R)$.

(7) Let K be a field. Show that $K[\![X_1, \ldots, X_n]\!]$ is a domain of dimension n.