LYING OVER: Let $R \subseteq S$ be an integral inclusion. Then the induced map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is surjective. That is, for any prime $\mathfrak{p} \in \operatorname{Spec}(R)$, there is a prime $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$; i.e., a prime *lying over* $\mathfrak{p}$.

INCOMPARABILITY: Let $R \to S$ be integral (but not necessarily injective). Then for any $\mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Spec}(S)$ such that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R$, we have $\mathfrak{q}_1 \not\subseteq \mathfrak{q}_2$. That is, any two primes lying over the same prime are *incomparable*.

GOING UP: Let $R \to S$ be integral (but not necessarily injective). Then for any $\mathfrak{p} \subsetneq \mathfrak{P}$ in $\operatorname{Spec}(R)$ and $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, there is some $\mathfrak{Q} \in \operatorname{Spec}(S)$ such that $\mathfrak{q} \subseteq \mathfrak{Q}$ and $\mathfrak{Q} \cap R = \mathfrak{P}$.

GOING DOWN: Let $R \subseteq S$ be an integral inclusion of domains, and assume that R is normal. Then for any $\mathfrak{p} \subsetneq \mathfrak{P}$ in $\operatorname{Spec}(R)$ and $\mathfrak{Q} \in \operatorname{Spec}(S)$ such that $\mathfrak{Q} \cap R = \mathfrak{P}$, there is some $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $\mathfrak{q} \subseteq \mathfrak{Q}$ and $\mathfrak{q} \cap R = \mathfrak{p}$.

LEMMA: Let $R \subseteq S$ be an integral inclusion and I an ideal of R. Then any element of $s \in IS$ satisfies a monic equation over R of the form[1]

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0 \qquad \text{with } a_i \in I \text{ for all } i.$$

(1) Proof of Lying Over from the Lemma: Let $R \subseteq S$ be an integral inclusion.
- **(a)** Use the Lemma to show that if $\mathfrak{p}$ is prime, then $\mathfrak{p}S \cap R = \mathfrak{p}$.
- **(b)** Show that $(R \smallsetminus \mathfrak{p})^{-1}(S/\mathfrak{p}S)$ is not the zero "ring".
- **(c)** Deduce[2] the Theorem.

(2) Proof of Lemma: Let $R \subseteq S$ be an integral inclusion and I an ideal of R.
- **(a)** Show that if $s \in IS$, then there is a module-finite R-subalgebra of S, say T, such that $s \in IT$, so we can assume that S is module-finite.
- **(b)** Write $S = \sum_i Rs_i$ and $v = [s_1, \ldots, s_t]$. Show that there is some $t \times t$ matrix A with entries in I such that $rv = vA$.
- **(c)** Apply a TRICK and conclude the proof.

(3) Proof of Incomparability: Let $R \to S$ be integral.
- **(a)** Explain[3] why the Theorem is true when R is a field.
- **(b)** Let $\mathfrak{p}$ in $\operatorname{Spec}(R)$. Use the definition to explain why the map $R/\mathfrak{p} \to S/\mathfrak{p}S$ is integral, and why the map $(R \smallsetminus \mathfrak{p})^{-1}(R/\mathfrak{p}) \to (R \smallsetminus \mathfrak{p})^{-1}(S/\mathfrak{p}S)$ is integral.
- **(c)** Use the previous parts (plus an old bijection) to prove the Theorem.

(4) Proof of Going Up: Show that $R/\mathfrak{p} \to S/\mathfrak{q}$ is an integral inclusion, apply Lying Over, and deduce the Theorem.

[1]In fact, one can take $a_i \in I^i$ for each i by the same proof, which is often useful.

[2]The old bijection $\operatorname{Spec}(W^{-1}(T/J)) \longleftrightarrow \{\mathfrak{q} \in \operatorname{Spec}(T) \mid \mathfrak{q} \cap W = \varnothing \text{ and } J \subseteq \mathfrak{q}\}$ may come in handy.

[3]Hint: Recall an old fact about integral extensions of domains...

(5) Proof of Going Down.

(a) Explain why it suffices to show that $(S \smallsetminus \mathfrak{Q})(R \smallsetminus \mathfrak{p}) \cap \mathfrak{p}S$ is empty.

(b) Let x be an element of the intersection. Show that[4] the minimal monic polynomial $f(x)$ of x over $\mathrm{Frac}(R)$ has all nonleading coefficients in $\mathfrak{p}$.

(c) Write $x = rs$ with $r \in R \smallsetminus \mathfrak{p}$ and $s \in S \smallsetminus \mathfrak{Q}$. Show that $g(s) = f(rs)/r^n$ is the minimal polynomial of s over $\mathrm{Frac}(R)$.

(d) Show that $g(s)$ has coefficients in R, and obtain a contradiction to the assumption that x was an element of the intersection.

(6) (a) Show that if S is module-finite over R with t generators, then for every $\mathfrak{p} \in \mathrm{Spec}(R)$, at most t distinct primes of S contract to $\mathfrak{p}$.

(b) Give an example of an integral inclusion $R \subseteq S$ such that there are primes of R with arbitrarily many primes contracting to it.

[4]Hint: First show all the coefficients are in R. For this, note that every coefficient of the minimal polynomial is a polynomial expression of the roots of f in an algebraic closure of $\mathrm{Frac}(R)$.