LYING OVER: Let $R \subseteq S$ be an integral inclusion. Then the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective. That is, for any prime $\mathfrak{p} \in \mathrm{Spec}(R)$, there is a prime $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$; i.e., a prime *lying over* $\mathfrak{p}$.

INCOMPARABILITY: Let $R \to S$ be integral (but not necessarily injective). Then for any $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(S)$ such[1] that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R$, we have $\mathfrak{q}_1 \not\subsetneq \mathfrak{q}_2$. That is, any two primes lying over the same prime are *incomparable*.

GOING UP: Let $R \to S$ be integral (but not necessarily injective). Then for any $\mathfrak{p} \subsetneq \mathfrak{P}$ in $\mathrm{Spec}(R)$ and $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, there is some $\mathfrak{Q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \subseteq \mathfrak{Q}$ and $\mathfrak{Q} \cap R = \mathfrak{P}$.

GOING DOWN: Let $R \subseteq S$ be an integral inclusion of domains, and assume that R is normal. Then for any $\mathfrak{p} \subsetneq \mathfrak{P}$ in $\mathrm{Spec}(R)$ and $\mathfrak{Q} \in \mathrm{Spec}(S)$ such that $\mathfrak{Q} \cap R = \mathfrak{P}$, there is some $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \subseteq \mathfrak{Q}$ and $\mathfrak{q} \cap R = \mathfrak{p}$.

COROLLARY: Let $R \to S$ be integral.

(1) If S is Noetherian, then for any $\mathfrak{p} \in \mathrm{Spec}(R)$, the set of primes in S that contract to $\mathfrak{p}$ is finite.
(2) If $R \subseteq S$ is an inclusion, and S is Noetherian, then for any $\mathfrak{p} \in \mathrm{Spec}(R)$, the set of primes in S that contract to $\mathfrak{p}$ is nonempty and finite.
(3) For any $\mathfrak{q} \in \mathrm{Spec}(S)$, we have $\mathrm{height}(\mathfrak{q}) \leq \mathrm{height}(\mathfrak{q} \cap R)$.
(4) $\dim(S) \leq \dim(R)$.
(5) If $R \subseteq S$ is an inclusion, then $\dim(R) = \dim(S)$.
(6) If $R \subseteq S$ is an inclusion, R is a normal domain, and S is a domain, then for any $\mathfrak{q} \in \mathrm{Spec}(S)$, we have $\mathrm{height}(\mathfrak{q}) = \mathrm{height}(\mathfrak{q} \cap R)$.

(1) Hypotheses of Lying Over and Incomparability:
- **(a)** Consider the inclusion map $\mathbb{Z} \subseteq \mathbb{Q}$. Show that the conclusion of Lying Over fails. Which hypotheses are true?
- **(b)** Consider the quotient map $\mathbb{C}[X] \to \mathbb{C}[X]/(X) \cong \mathbb{C}$. Show that the conclusion of Lying Over fails. Which hypotheses are true?
- **(c)** Consider the inclusion map $\mathbb{C} \subseteq \mathbb{C}[X]$. Show that the conclusion of Incomparability fails. Which hypotheses are true?
- **(d)** Consider the inclusion map $R := \mathbb{C}[X^2] \subseteq S := \mathbb{C}[X]$. Describe all of the primes $\mathfrak{q}_i$ that contract to $\mathfrak{p} := (X^2 - 1)R$. Verify the conclusions on Incomparability and Lying Over for $\mathfrak{p}$ and the $\mathfrak{q}_i$.

[1]Reminder: by abuse of notation, even when $\phi : R \to S$ is not injective, we write $\mathfrak{q} \cap R$ for $\phi^{-1}(\mathfrak{q}) \subseteq R$.

(a) The prime $2\mathbb{Z}$ is not the contraction of any prime; the only prime in the image is $0\mathbb{Z}$. This is an inclusion but not integral.
(b) The prime (0) is not in the image, because the contraction of every ideal contains (X). This is integral, but not an inclusion.
(c) Both (0) and (X) in $\mathbb{C}[X]$ contract to (0) in $\mathbb{C}$, but $(0) \subsetneqq (X)$.
(d) A prime that contracts to (X^2-1) must contain X^2-1, and hence must contain $X-1$ or $X+1$. We find that $\mathfrak{q}_1 = (X-1)$ and $\mathfrak{q}_2 = (X+1)$ both contract to (X^2-1) in R. In particular, something contracts to $\mathfrak{p}$, so Lying Over holds, and the two primes that do are incomparable, so Incomparability holds.

(2) Proof of Corollary using the theorems: Let $R \to S$ be integral.
(a) Use one of the Theorems above to show that for any chain of primes

$$\mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_n = \mathfrak{q} \qquad \text{in } \operatorname{Spec}(S)$$

the containments

$$(\mathfrak{q}_0 \cap R) \subseteq (\mathfrak{q}_1 \cap R) \subseteq \cdots \subseteq (\mathfrak{q}_n \cap R) = (\mathfrak{q} \cap R) \quad \text{in } \operatorname{Spec}(R)$$

are proper. Explain why this implies Part (3).
(b) Deduce part (4) from part (3).
(c) Let $R \subseteq S$ be an inclusion, and take a chain of primes

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n \qquad \text{in } \operatorname{Spec}(R).$$

Use Lying Over and Going up to find a chain of primes

$$\mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_n \qquad \text{in } \operatorname{Spec}(S)$$

such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for all i. Deduce part (5).
(d) Prove part (6).
(e) Let $\mathfrak{q} \in \operatorname{Spec}(S)$ and $\mathfrak{p} \in \operatorname{Spec}(R)$. Show that if $\mathfrak{q} \cap R = \mathfrak{p}$, then $\mathfrak{q} \supseteq \mathfrak{p}S$, and if $\mathfrak{q}_0$ is some prime of S such that $\mathfrak{p}S \subseteq \mathfrak{q}_0 \subseteq \mathfrak{q}$, then $\mathfrak{q}_0 \cap R = \mathfrak{p}$ also.
(f) Show that every prime that contracts to $\mathfrak{p}$ is a minimal prime of $\mathfrak{p}S$, and deduce parts (1) and (2).

(a) These containments are proper by incomparability. If the height of $\mathfrak{q}$ is at least n, then there is a proper chain as above, and then there is a proper chain of primes up to $\mathfrak{q} \cap R$ of length n, so the height of $\mathfrak{q} \cap R$ is at least n.
(b) If the dimension of S is at least n, then there is a prime of height at least n in $\operatorname{Spec}(S)$, so there is a prime of height at least n in $\operatorname{Spec}(R)$, and the dimension of R is at least n.
(c) By Lying Over we can take a $\mathfrak{q}_0$ that contracts to $\mathfrak{p}_0$. Applying Going up, we get a prime $\mathfrak{q}_1$ that contains $\mathfrak{q}_0$ and contracts to $\mathfrak{p}_1$. Continuing like so, we build the chain as required. Thus, if the dimension of R is at least n, there is a chain in $\operatorname{Spec}(S)$ of length at least n, so $\dim(S)$ is at least n. Thus, $\dim(R) \leq \dim(S)$.
(d) Take $\mathfrak{q} \in \operatorname{Spec}(S)$ and $\mathfrak{p} \in \operatorname{Spec}(R)$ and a chain of primes in $\operatorname{Spec}(R)$ of length n with $\mathfrak{p}_n = \mathfrak{p}$. We can apply Going Down to find a $\mathfrak{q}_{n-1} \in \operatorname{Spec}(S)$ such that $\mathfrak{q}_{n-1} \subsetneqq \mathfrak{q}_n$ such that $\mathfrak{q}_{n-1} \cap R = \mathfrak{p}_{n-1}$. Continuing like so, we can form a chain

of primes in $\operatorname{Spec}(S)$ of length n. This implies that the height of $\mathfrak{q} \cap R$ is less than or equal to the height of $\mathfrak{q}$.

(e) By definition, any ideal of S that contains the image of $\mathfrak{p}$ contains $\mathfrak{p}S$, so $\mathfrak{q} \cap R \supseteq \mathfrak{p}$ if and only if $\mathfrak{q} \supseteq \mathfrak{p}S$. In particular, $\mathfrak{q}_0 \cap R \supseteq \mathfrak{p}S$ implies $\mathfrak{q}_0 \cap R \supseteq \mathfrak{p}$ and $\mathfrak{q}_0 \cap R \subseteq \mathfrak{q} \cap R = \mathfrak{p}$, so $\mathfrak{q}_0 \cap R = \mathfrak{p}$.

(f) If $\mathfrak{q}$ contracts to $\mathfrak{p}$, then $\mathfrak{q}$ contains a minimal prime of $\mathfrak{p}S$ that contracts to $\mathfrak{p}$ by the previous part. Then by Incomparability, $\mathfrak{q}$ is a minimal prime of $\mathfrak{p}S$. By Noetherianity, since $\mathfrak{p}S$ has finitely many minimal primes, there are at most finitely many primes that contract to $\mathfrak{p}$, showing (1). Finally, (2) follows from (1) and Lying Over.

(3) Hypotheses of Going Down:

  (a) Consider the inclusion map $\mathbb{C}[X] \subseteq \mathbb{C}[X,Y]/(XY, Y^2 - Y)$. Show that[2] the conclusion of Going Down fails. Which hypotheses are true?

  (b) Consider the inclusion map $\mathbb{C}[X(1-X), X^2(1-X), Y, XY] \subseteq \mathbb{C}[X,Y]$. Show that[3] the conclusion of Going Down fails. Which hypotheses are true?

(a) Let $R = K[X] \subseteq S = K[X,Y]/(XY, Y^2 - Y)$. R is a normal domain, and the inclusion is integral: $y^2 - y = 0$ is an integral dependence relation for y over R, so S is generated by one integral element. Now, $(1-y)$ is a minimal prime of S: $y \in S \smallsetminus (1-y)$, so x goes to zero in the localization (since $xy = 0$) and $1-y$ goes to zero in the localization (since $y(1-y) = 0$), so the localization is a copy of K, which has only one prime, (0). We have $x = x - xy = x(1-y) \in (1-y)$, so the contraction contains (X), so must be (X). But, by minimality, we can't "go down" from $(1-y)$ to a prime lying over (0).

(b) The element X is integral over R: $X(1-X) \in R$ is a recipe: X is a root of $T^2 - T - X(1-X)$. Note that X is in the fraction field of R, so this element shows both that S is integral over R, and that R is not normal. Now, $\mathfrak{q} = (1-X, Y) \subseteq S$ is a maximal ideal lying over the maximal ideal $\mathfrak{p} = (X(1-X), X^2(1-X), Y, XY)$ in R. We have $xS \cap R = (X(1-X), X^2(1-X), XY)R = \mathfrak{p}'$, but we claim that no prime contained in $\mathfrak{q}$ lies over $\mathfrak{p}'$. Such a prime must contain $X(1-X)$ and XY, but not X (this would make it the unit ideal), so must contain Y and $1-X$, and the contraction is then $\mathfrak{p}$, which is too big!

[2] Consider $(1-y)$, (X), and (0).

[3] Consider $(1-X, Y)$, $(X(1-X), X^2(1-X), Y, XY)$, and $(1-X, Y) \cap R$.