§7.30: COHEN-SEIDENBERG THEOREMS: APPLICATIONS

LYING OVER: Let $R \subseteq S$ be an integral inclusion. Then the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective. That is, for any prime $\mathfrak{p} \in \mathrm{Spec}(R)$, there is a prime $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$; i.e., a prime *lying over* $\mathfrak{p}$.

INCOMPARABILITY: Let $R \to S$ be integral (but not necessarily injective). Then for any $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(S)$ such[1] that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R$, we have $\mathfrak{q}_1 \not\subseteq \mathfrak{q}_2$. That is, any two primes lying over the same prime are *incomparable*.

GOING UP: Let $R \to S$ be integral (but not necessarily injective). Then for any $\mathfrak{p} \subsetneqq \mathfrak{P}$ in $\mathrm{Spec}(R)$ and $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, there is some $\mathfrak{Q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \subseteq \mathfrak{Q}$ and $\mathfrak{Q} \cap R = \mathfrak{P}$.

GOING DOWN: Let $R \subseteq S$ be an integral inclusion of domains, and assume that R is normal. Then for any $\mathfrak{p} \subsetneqq \mathfrak{P}$ in $\mathrm{Spec}(R)$ and $\mathfrak{Q} \in \mathrm{Spec}(S)$ such that $\mathfrak{Q} \cap R = \mathfrak{P}$, there is some $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $\mathfrak{q} \subseteq \mathfrak{Q}$ and $\mathfrak{q} \cap R = \mathfrak{p}$.

COROLLARY: Let $R \to S$ be integral.

(1) If S is Noetherian, then for any $\mathfrak{p} \in \mathrm{Spec}(R)$, the set of primes in S that contract to $\mathfrak{p}$ is finite.
(2) If $R \subseteq S$ is an inclusion, and S is Noetherian, then for any $\mathfrak{p} \in \mathrm{Spec}(R)$, the set of primes in S that contract to $\mathfrak{p}$ is nonempty and finite.
(3) For any $\mathfrak{q} \in \mathrm{Spec}(S)$, we have $\mathrm{height}(\mathfrak{q}) \leq \mathrm{height}(\mathfrak{q} \cap R)$.
(4) $\dim(S) \leq \dim(R)$.
(5) If $R \subseteq S$ is an inclusion, then $\dim(R) = \dim(S)$.
(6) If $R \subseteq S$ is an inclusion, R is a normal domain, and S is a domain, then for any $\mathfrak{q} \in \mathrm{Spec}(S)$, we have $\mathrm{height}(\mathfrak{q}) = \mathrm{height}(\mathfrak{q} \cap R)$.

(1) Hypotheses of Lying Over and Incomparability:
- **(a)** Consider the inclusion map $\mathbb{Z} \subseteq \mathbb{Q}$. Show that the conclusion of Lying Over fails. Which hypotheses are true?
- **(b)** Consider the quotient map $\mathbb{C}[X] \to \mathbb{C}[X]/(X) \cong \mathbb{C}$. Show that the conclusion of Lying Over fails. Which hypotheses are true?
- **(c)** Consider the inclusion map $\mathbb{C} \subseteq \mathbb{C}[X]$. Show that the conclusion of Incomparability fails. Which hypotheses are true?
- **(d)** Consider the inclusion map $R := \mathbb{C}[X^2] \subseteq S := \mathbb{C}[X]$. Describe all of the primes $\mathfrak{q}_i$ that contract to $\mathfrak{p} := (X^2 - 1)R$. Verify the conclusions on Incomparability and Lying Over for $\mathfrak{p}$ and the $\mathfrak{q}_i$.

[1]Reminder: by abuse of notation, even when $\phi : R \to S$ is not injective, we write $\mathfrak{q} \cap R$ for $\phi^{-1}(\mathfrak{q}) \subseteq R$.

(2) Proof of Corollary using the theorems: Let $R \to S$ be integral.

(a) Use one of the Theorems above to show that for any chain of primes

$$\mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_n = \mathfrak{q} \qquad \text{in } \operatorname{Spec}(S)$$

the containments

$$(\mathfrak{q}_0 \cap R) \subseteq (\mathfrak{q}_1 \cap R) \subseteq \cdots \subseteq (\mathfrak{q}_n \cap R) = (\mathfrak{q} \cap R) \quad \text{in } \operatorname{Spec}(R)$$

are proper. Explain why this implies Part (3).

(b) Deduce part (4) from part (3).

(c) Let $R \subseteq S$ be an inclusion, and take a chain of primes

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n \qquad \text{in } \operatorname{Spec}(R).$$

Use Lying Over and Going up to find a chain of primes

$$\mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_n \qquad \text{in } \operatorname{Spec}(S)$$

such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for all i. Deduce part (5).

(d) Prove part (6).

(e) Let $\mathfrak{q} \in \operatorname{Spec}(S)$ and $\mathfrak{p} \in \operatorname{Spec}(R)$. Show that if $\mathfrak{q} \cap R = \mathfrak{p}$, then $\mathfrak{q} \supseteq \mathfrak{p}S$, and if $\mathfrak{q}_0$ is some prime of S such that $\mathfrak{p}S \subseteq \mathfrak{q}_0 \subseteq \mathfrak{q}$, then $\mathfrak{q}_0 \cap R = \mathfrak{p}$ also.

(f) Show that every prime that contracts to $\mathfrak{p}$ is a minimal prime of $\mathfrak{p}S$, and deduce parts (1) and (2).

(3) Hypotheses of Going Down:

(a) Consider the inclusion map $\mathbb{C}[X] \subseteq \mathbb{C}[X,Y]/(XY, Y^2 - Y)$. Show that[2] the conclusion of Going Down fails. Which hypotheses are true?

(b) Consider the inclusion map $\mathbb{C}[X(1-X), X^2(1-X), Y, XY] \subseteq \mathbb{C}[X,Y]$. Show that[3] the conclusion of Going Down fails. Which hypotheses are true?

[2]Consider $(1-y)$, (X), and (0).

[3]Consider $(1-X, Y)$, $(X(1-X), X^2(1-X), Y, XY)$, and $(1-X, Y) \cap R$.