DEFINITION: Let R be a ring.

- A **chain of primes of length** n is
$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n \qquad \text{with } \mathfrak{p}_i \in \operatorname{Spec}(R).$$
We may say this chain is **from** $\mathfrak{p}_0$ and/or **to** $\mathfrak{p}_n$ to indicate the minimal and/or maximal elements.
- A chain of primes as above is **saturated** if for each i, there is no prime $\mathfrak{q}$ such that $\mathfrak{p}_i \subsetneqq \mathfrak{q} \subsetneqq \mathfrak{p}_{i+1}$.
- The **dimension** of R is
$$\dim(R) := \sup\{n \geq 0 \mid \text{there is a chain of primes of length } n \text{ in } \operatorname{Spec}(R)\}.$$
- The **height** of a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is
$$\operatorname{height}(\mathfrak{p}) := \sup\{n \geq 0 \mid \text{there is a chain of primes to } \mathfrak{p} \text{ of length } n \text{ in } \operatorname{Spec}(R)\}.$$
- The **height** of an arbitrary proper ideal $I \subseteq R$ is
$$\operatorname{height}(I) := \inf\{\operatorname{height}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min}(I)\}.$$

(1) Let K be field. Use the definition of dimension to prove the following:

(a) $\dim(K) = 0$.
(b) If R is a PID, but not a field, then $\dim(R) = 1$.
(c) $\dim(K[X_1, \dots, X_n]) \geq n$.
(d) $\dim(K[\![X_1, \dots, X_n]\!]) \geq n$.
(e) $\dim(K[X_1, X_2, X_3, \dots]) = \infty$.

(a) The only prime is (0) so every chain has length zero.
(b) Every nonzero prime is maximal, so the longest chains have length one.
(c) There is a chain $(0) \subsetneqq (X_1) \subsetneqq (X_1, X_2) \subsetneqq \cdots \subsetneqq (X_1, \dots, X_n)$.
(d) Same as above.
(e) Same as above by keep going.

(2) Let R be a ring, I an ideal, and $\mathfrak{p}$ a prime ideal. Use the definitions to prove the following:

(a) $\operatorname{height}(\mathfrak{p}) = 0$ if and only if $\mathfrak{p} \in \operatorname{Min}(R)$.
(b) $\operatorname{height}(I) = 0$ if and only if $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Min}(R)$.
(c) If R is a domain and $I \neq 0$, then $\operatorname{height}(I) > 0$.
(d) $\dim(R/\mathfrak{p}) = \sup\{n \geq 0 \mid \text{there is a chain of primes of length } n \text{ in } V(\mathfrak{p})\}$.
(e) $\dim(R/I) = \sup\{n \geq 0 \mid \text{there is a chain of primes of length } n \text{ in } V(I)\}$.
(f) If R is a domain and $I \neq 0$, and $\dim(R) < \infty$, then $\dim(R/I) < \dim(R)$.
(g) $\dim(R) = \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min}(R)\}$.
(h) $\dim(R_\mathfrak{p}) = \operatorname{height}(\mathfrak{p})$.
(i) $\dim(R) = \sup\{\dim(R_\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Max}(R)\}$.
(j) $\operatorname{height}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \sup\left\{ n \geq 0 \;\middle|\; \begin{array}{l} \text{there is a chain of primes of length } n \\ \text{in } \operatorname{Spec}(R) \text{ such that } \mathfrak{p}_i = \mathfrak{p} \text{ for some } i \end{array} \right\}$
(k) $\operatorname{height}(\mathfrak{p}) + \dim(R/\mathfrak{p}) \leq \dim(R)$.
(l) $\operatorname{height}(I) + \dim(R/I) \leq \sup\left\{ n \geq 0 \;\middle|\; \begin{array}{l} \text{there is a chain of primes of length } n \\ \text{in } \operatorname{Spec}(R) \text{ such that } \mathfrak{p}_i \in \operatorname{Min}(I) \text{ for some } i \end{array} \right\}$.
(m) $\operatorname{height}(I) + \dim(R/I) \leq \dim(R)$.

(a) Height zero means it can't contain any other primes, because that would be a recipe for a chain of positive length.
(b) Height zero means some minimal prime of it is a minimal prime of R. That is the same as being contained in a minimal prime of R.
(c) The only minimal prime of a domain is zero; see above.
(d) Primes in $R/\mathfrak{p}$ correspond to primes of R containing $\mathfrak{p}$.
(e) Primes of R/I correspond to primes of R containing I.
(f) If R is a domain and $I \neq 0$, then any prime in $V(I)$ properly contains zero, so a chain in $V(I)$ can be made one longer by throwing in (0) at the bottom.
(g) $(\geq)$ is clear since $V(\mathfrak{p}) \subseteq \mathrm{Spec}(R)$. $(\leq)$ follows since any chain of primes in R can be extended to a chain from a minimal prime.
(h) Primes in $R_{\mathfrak{p}}$ correspond to primes of R that are contained in $\mathfrak{p}$; thus any chain of primes to a prime contained in $\mathfrak{p}$ corresponds to a chain of primes in $R_{\mathfrak{p}}$ and conversely.
(i) $(\geq)$ is clear since $\Lambda(\mathfrak{m}) \subseteq \mathrm{Spec}(R)$. $(\leq)$ follows since any chain of primes in R can be extended to a chain to a maximal ideal.
(j) As above, we identify chains of primes in $R/\mathfrak{p}$ with chains in $V(\mathfrak{p})$. For $(\geq)$, given such a chain, break it at $\mathfrak{p}$ to get a chain to $\mathfrak{p}$ and a chain from $\mathfrak{p}$; the first has length at most $\mathrm{height}(\mathfrak{p})$ and the second has length at most $\dim(R/\mathfrak{p})$. For $(\leq)$, given a chain of primes to $\mathfrak{p}$ and a chain in $V(\mathfrak{p})$, we obtain by concatenation a chain in R whose length is at least the sum of the lengths.
(k) Clear from the previous.
(l) For $(\leq)$, if $\mathrm{height}(I) \geq a$ and $\dim(R/I) \geq b$, then for every $\mathfrak{p} \in \mathrm{Min}(I)$, there is a chain of primes of $\mathfrak{p}$ of length at least a, and there exists $\mathfrak{p}_0 \in \mathrm{Min}(I)$ and a chain of primes from $\mathfrak{p}_0$ of length b. Concatenating, we get a chain of primes through $\mathfrak{p}_0$ of length at least $a+b$. This shows the inequality.
(m) Clear from the previous.

(3) Dimension vs height
   (a) Let K be a field and $R = K[X,Y,Z]/(XY,XZ)$. Let $\mathfrak{p} = (y,z)$. Compute $\dim(R/\mathfrak{p})$ and $\mathrm{height}(\mathfrak{p})$, and show that $\dim(R) \geq 2$.
   (b) Let $R = \mathbb{Z}_{(2)}[X]$. Let $\mathfrak{p} = (2X-1)$. Compute $\dim(R/\mathfrak{p})$ and[1] $\mathrm{height}(\mathfrak{p})$, and show that $\dim(R) \geq 2$.

(a) $R/\mathfrak{p} \cong K[X]$ so its dimension is 1. $\mathfrak{p}$ is minimal so its height is 0. But $(x) \subseteq (x,y) \subseteq (x,y,z)$ shows that $\dim(R) \geq 2$.
(b) $R/\mathfrak{p} \cong \mathbb{Z}_{(2)}[1/2] \cong \mathbb{Q}$ so $\dim(R/\mathfrak{p}) = 0$. $\mathfrak{p}$ has height 1 since R is a UFD; see below. But R has dimension at least 2 since one has $(0) \subseteq (2) \subseteq (2,X)$.

(4) Let R be a domain. Show that R is a UFD if and only if every prime ideal of height one is principal.

This solution is embargoed.

(5) Does it follow from the definition that in a Noetherian ring, every prime has finite height?

No, there could be distinct chains that get longer and longer.

[1] You can use the next problem if you like.

(6) In this problem we will construct a Noetherian ring of infinite dimension. Let K be a field, $S = K[X_{1,1}, X_{2,1}, X_{2,2}, X_{3,1}, X_{3,2}, X_{3,3}, \dots]$, and $W = S \smallsetminus \bigcup_t (X_{t,1}, \dots, X_{t,t})$.

(a) Let A be a ring. Suppose that $\mathrm{Max}(A)$ is finite, $A_{\mathfrak{m}}$ is Noetherian for every $\mathfrak{m} \in \mathrm{Max}(A)$, and every nonzero element is contained in finitely many maximal ideals. Show that A is Noetherian.

(b) Let $\mathfrak{p}_t = (X_{t,1}, \dots, X_{t,t,})$ for $t \geq 1$. Let I be an ideal. Show that if $I \subseteq \bigcup_{t \geq 1} \mathfrak{p}_t$, then there is[2] some $t \geq 1$ such that $I \subseteq \mathfrak{p}_t$.

(c) Show that $R := W^{-1}S$ is Noetherian and infinite dimensional.

(a) Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending chain of ideals; without loss of generality, I_1 is nonzero. By hypothesis, $V_{\max}(I_1)$ is finite, and $V_{\max}(I_i) \supseteq V_{\max}(I_{i+1})$ for every i by definition. A descending chain of finite sets stabilizes, so $X = V_{\max}(I_i)$ stabilizes. Then for each $\mathfrak{m} \in X$, the chain

$$(I_1)_{\mathfrak{m}} \subseteq (I_2)_{\mathfrak{m}} \subseteq (I_3)_{\mathfrak{m}} \subseteq \cdots$$

stabilizes. In particular, there is some t such that $(I_i)_{\mathfrak{m}} = (I_{i+1})_{\mathfrak{m}}$ for all $i \geq t$ and all maximal ideals containing I_{i+1}. Thus, $\mathrm{Supp}(I_{i+1}/I_i)$ contains no maximal ideals, hence is empty, so $I_i = I_{i+1}$ for all $i \geq t$; i.e., the chain stabilizes.

(b) If $I = 0$ this is clear, so suppose $I \neq 0$, that $I \subseteq \bigcup_{i \in \mathbb{N}} \mathfrak{p}_i$. For $s \in S$, set

$$v(s) := \{i \mid f \in \mathfrak{p}_i\}.$$

Since s involves finitely many variables, $v(s)$ is finite for each nonzero $s \in S$. Our hypothesis translates to saying $v(f)$ is nonempty for each $f \in I$.

We claim that for any $f, g \in I$, there is some $h \in I$ with $v(h) \subseteq v(f) \cap v(g)$. Namely, let k be larger than the first index of any variable in f or g, and t be an integer greater than the degree of f and set $h = f + x_k^t g$. Then f and $x_k^t g$ have no monomials in common (since the degrees of all the monomials in $x_k^t g$ are at least t and the degree of the monomials in f are all less than t) so none can cancel from each other. In particular, if x_ℓ divides h in T, then x_ℓ divides both f and $x_k^t g$ in T; i.e., $v(h) \subseteq v(f) \cap v(g)$ as claimed.

Thus, fixing some nonzero $f \in I$, for every $g \in I$, $v(f) \cap v(g)$ is nonempty. That means that every $g \in I$ is in some $\mathfrak{p}_i$ for $i \in v(f)$, so $I \subseteq \bigcup_{i \in v(f)} \mathfrak{p}_i$, which is a finite union of primes. By the usual version of prime avoidance, $I \subseteq \mathfrak{p}_i$ for some i.

(c) Clearly R is infinite dimensional, since for any n, there is a chain of primes contained in $\mathfrak{p}_n$ of length n, which yields a chain of primes of length n in R. To see that R is Noetherian, note first that by the previous part, any ideal of S that does not intersect W is contained in some $\mathfrak{p}_t$, so every ideal $W^{-1}R$ is contained in some $W^{-1}\mathfrak{p}_t$, so these are the maximal ideals of R. Now note that any element considered as a fraction has a numerator in at most finitely many $\mathfrak{p}_n$. Moreover, localizing at $\mathfrak{p}_t$ yields ring isomorphic to a localization of polynomial ring in t variables over a field, which is Noetherian. Thus, by the Lemma, R is Noetherian.

[2] Note that this looks similar to prime avoidance, but with an infinite set of primes. For $f \in S$, let $v(f) := \{t \mid f \in \mathfrak{p}_t\}$. Show that for any $f, g \in I$, there is some $h \in I$ with $v(h) \subseteq v(f) \cup v(g)$. Then apply prime avoidance.