DEFINITION: Let R be a ring.

- A **chain of primes of length** n is
$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n \qquad \text{with } \mathfrak{p}_i \in \operatorname{Spec}(R).$$
We may say this chain is **from** $\mathfrak{p}_0$ and/or **to** $\mathfrak{p}_n$ to indicate the minimal and/or maximal elements.
- A chain of primes as above is **saturated** if for each i, there is no prime $\mathfrak{q}$ such that $\mathfrak{p}_i \subsetneqq \mathfrak{q} \subsetneqq \mathfrak{p}_{i+1}$.
- The **dimension** of R is
$$\dim(R) := \sup\{n \geq 0 \mid \text{there is a chain of primes of length } n \text{ in } \operatorname{Spec}(R)\}.$$
- The **height** of a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is
$$\operatorname{height}(\mathfrak{p}) := \sup\{n \geq 0 \mid \text{there is a chain of primes to } \mathfrak{p} \text{ of length } n \text{ in } \operatorname{Spec}(R)\}.$$
- The **height** of an arbitrary proper ideal $I \subseteq R$ is
$$\operatorname{height}(I) := \inf\{\operatorname{height}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min}(I)\}.$$

(1) Let K be field. Use the definition of dimension to prove the following:
- **(a)** $\dim(K) = 0$.
- **(b)** If R is a PID, but not a field, then $\dim(R) = 1$.
- **(c)** $\dim(K[X_1, \dots, X_n]) \geq n$.
- **(d)** $\dim(K[\![X_1, \dots, X_n]\!]) \geq n$.
- **(e)** $\dim(K[X_1, X_2, X_3, \dots]) = \infty$.

(2) Let R be a ring, I an ideal, and $\mathfrak{p}$ a prime ideal. Use the definitions to prove the following:
- **(a)** $\operatorname{height}(\mathfrak{p}) = 0$ if and only if $\mathfrak{p} \in \operatorname{Min}(R)$.
- **(b)** $\operatorname{height}(I) = 0$ if and only if $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Min}(R)$.
- **(c)** If R is a domain and $I \neq 0$, then $\operatorname{height}(I) > 0$.
- **(d)** $\dim(R/\mathfrak{p}) = \sup\{n \geq 0 \mid \text{there is a chain of primes of length } n \text{ in } V(\mathfrak{p})\}$.
- **(e)** $\dim(R/I) = \sup\{n \geq 0 \mid \text{there is a chain of primes of length } n \text{ in } V(I)\}$.
- **(f)** If R is a domain and $I \neq 0$, and $\dim(R) < \infty$, then $\dim(R/I) < \dim(R)$.
- **(g)** $\dim(R) = \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min}(R)\}$.
- **(h)** $\dim(R_\mathfrak{p}) = \operatorname{height}(\mathfrak{p})$.
- **(i)** $\dim(R) = \sup\{\dim(R_\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Max}(R)\}$.
- **(j)** $\operatorname{height}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \sup\left\{ n \geq 0 \,\middle|\, \begin{array}{l} \text{there is a chain of primes of length } n \\ \text{in } \operatorname{Spec}(R) \text{ such that } \mathfrak{p}_i = \mathfrak{p} \text{ for some } i \end{array} \right\}$
- **(k)** $\operatorname{height}(\mathfrak{p}) + \dim(R/\mathfrak{p}) \leq \dim(R)$.
- **(l)** $\operatorname{height}(I) + \dim(R/I) \leq \sup\left\{ n \geq 0 \,\middle|\, \begin{array}{l} \text{there is a chain of primes of length } n \\ \text{in } \operatorname{Spec}(R) \text{ such that } \mathfrak{p}_i \in \operatorname{Min}(I) \text{ for some } i \end{array} \right\}$.
- **(m)** $\operatorname{height}(I) + \dim(R/I) \leq \dim(R)$.

(3) Dimension vs height
- (a) Let K be a field and $R = K[X, Y, Z]/(XY, XZ)$. Let $\mathfrak{p} = (y, z)$. Compute $\dim(R/\mathfrak{p})$ and $\operatorname{height}(\mathfrak{p})$, and show that $\dim(R) \geq 2$.
- (b) Let $R = \mathbb{Z}_{(2)}[X]$. Let $\mathfrak{p} = (2X - 1)$. Compute $\dim(R/\mathfrak{p})$ and[1] $\operatorname{height}(\mathfrak{p})$, and show that $\dim(R) \geq 2$.

(4) Let R be a domain. Show that R is a UFD if and only if every prime ideal of height one is principal.

[1]You can use the next problem if you like.

(5) Does it follow from the definition that in a Noetherian ring, every prime has finite height?

(6) In this problem we will construct a Noetherian ring of infinite dimension. Let K be a field, $S = K[X_{1,1}, X_{2,1}, X_{2,2}, X_{3,1}, X_{3,2}, X_{3,3}, \ldots]$, and $W = S \smallsetminus \bigcup_t (X_{t,1}, \ldots, X_{t,t})$.

(a) Let A be a ring. Suppose that $\mathrm{Max}(A)$ is finite, $A_{\mathfrak{m}}$ is Noetherian for every $\mathfrak{m} \in \mathrm{Max}(A)$, and every nonzero element is contained in finitely many maximal ideals. Show that A is Noetherian.

(b) Let $\mathfrak{p}_t = (X_{t,1}, \ldots, X_{t,t,})$ for $t \geq 1$. Let I be an ideal. Show that if $I \subseteq \bigcup_{t \geq 1} \mathfrak{p}_t$, then there is[2] some $t \geq 1$ such that $I \subseteq \mathfrak{p}_t$.

(c) Show that $R := W^{-1}S$ is Noetherian and infinite dimensional.

[2]Note that this looks similar to prime avoidance, but with an infinite set of primes. For $f \in S$, let $v(f) := \{t \mid f \in \mathfrak{p}_t\}$. Show that for any $f, g \in I$, there is some $h \in I$ with $v(h) \subseteq v(f) \cup v(g)$. Then apply prime avoidance.