DEFINITION: A **minimal primary decomposition** of an ideal I is a primary decomposition

$$I = Q_1 \cap \cdots \cap Q_n$$

such that $Q_i \not\supseteq \bigcap_{j\neq i} Q_j$, and $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$.

THEOREM (FIRST UNIQUENESS THEOREM FOR PRIMARY DECOMPOSITION): Let R be a Noetherian ring and I an ideal. Let

$$I = Q_1 \cap \cdots \cap Q_n$$

be a minimal primary decomposition of I. Then

$$\{\sqrt{Q_1}, \dots, \sqrt{Q_n}\} = \mathrm{Ass}_R(R/I).$$

In particular, the set of primes occurring as the radicals of the primary components are uniquely determined.

THEOREM (SECOND UNIQUENESS THEOREM FOR PRIMARY DECOMPOSITION): Let R be a Noetherian ring and I an ideal. Let

$$I = Q_1 \cap \cdots \cap Q_n$$

be a minimal primary decomposition of I. Suppose that $\mathfrak{p} = \sqrt{Q_i}$ is a *minimal* prime of I. Then $Q_i = IR_{\mathfrak{p}} \cap R$. In particular, the primary components corresponding to the minimal primes are uniquely determined.

LEMMA: Let $I_1, \dots, I_t$ be ideals. Then

(1) for any multiplicatively closed set W, $W^{-1}(I_1 \cap \cdots \cap I_t) = W^{-1}I_1 \cap \cdots \cap W^{-1}I_t$.
(2) $\mathrm{Ass}_R\left(R/\bigcap_{i=1}^t I_i\right) \subseteq \bigcup_{i=1}^t \mathrm{Ass}_R(R/I_i)$.

(1) Uniqueness theorems:

(a) Let K be a field, $R = K[X,Y]$ a polynomial ring, and $I = (X^2, XY)$. Verify[1] that $I = (X) \cap (X^2, Y) = (X) \cap (X^2, XY, Y^2)$ gives two different minimal primary decompositions of I.

(b) In the previous part, which aspects of the decomposition are the same, and which are different. Compare with the uniqueness theorems.

(c) Use the uniqueness theorems to explain why, for $n \in \mathbb{Z}$ with prime factorization $n = \pm p_1^{e_1} \cdots p_m^{e_m}$, the *only*[2] minimal primary decomposition of (n) is

$$(n) = (p_1^{e_1}) \cap \cdots \cap (p_m^{e_m}).$$

(a) (X) is prime, hence primary. (X^2, Y) and (X^2, XY, Y^2) both have radical (X, Y), which is maximal, so they are primary. In each case we have different radicals and neither component contained in the other.

[1] You can take for granted that in each case the intersection is I, but explain why the ideals are primary and the minimality hypotheses hold.

[2] We don't care about the order.

(b) In both cases the radicals of the primes are the same, and the (X)-component are the same.

(c) For any such decomposition, the prime ideals occurring are the same, since each prime is minimal, the the components are the same.

(2) Minimal primary decompositions: Let R be a Noetherian ring.

(a) Use the Lemma to explain why a finite intersection of $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary.

(b) Explain how to turn a general $I = Q_1 \cap \cdots \cap Q_m$ primary decomposition into a minimal primary decomposition.

(a) Because $\mathfrak{p}$-primary is equivalent to $\mathrm{Ass}_R(R/I) = \{\mathfrak{p}\}$.

(b) Intersect all of the Q_i's with the same radical to get a decomposition satisfying the second condition. Then remove any component that is contained in the intersection of the others to satisfy the first condition.

(3) Proof of Second Uniqueness Theorem:

(a) Use the definition of primary to show that if Q is $\mathfrak{p}$-primary, then $QR_{\mathfrak{p}} \cap R = Q$.

(b) Show[3] that if Q is $\mathfrak{q}$-primary and $\mathfrak{q} \not\subseteq \mathfrak{p}$, then $QR_{\mathfrak{p}} = R_{\mathfrak{p}}$.

(c) Let R be Noetherian and $I = Q_1 \cap \cdots \cap Q_n$ be a minimal primary decomposition, and $\mathfrak{p} = \sqrt{Q_i}$ a minimal prime of I. Use the Lemma to show that $IR_{\mathfrak{p}} = Q_i R_{\mathfrak{p}}$.

(d) Complete the proof.

(a) Clearly $Q \subseteq QR_{\mathfrak{p}} \cap R$. Let $r \in QR_{\mathfrak{p}} \cap R$, so there is some $q \in Q$ and $w \notin \mathfrak{p}$ such that $\frac{q}{w} = \frac{r}{1} \in R_{\mathfrak{p}}$. This means there is some $v \notin \mathfrak{p}$ such that $v(q - rw) = 0$ in R; i.e., $vwr = qv$, so in particular there is some $u \notin \mathfrak{p}$ such that $ur \in Q$. By definition of primary, $r \in Q$.

(b) We have $\mathrm{Supp}(R/Q) = V(Q) = V(\mathfrak{q})$. If $\mathfrak{p} \not\subseteq \mathfrak{q}$, then $\mathfrak{p} \notin V(\mathfrak{q})$, so $(R/Q)_{\mathfrak{p}} = 0$ and $R_{\mathfrak{p}} = QR_{\mathfrak{p}}$.

(c) We have $IR_{\mathfrak{p}} = Q_1 R_{\mathfrak{p}} \cap \cdots \cap Q_n R_{\mathfrak{p}}$. By the previous part, each term on the right is all of $R_{\mathfrak{q}}$ except $Q_i R_{\mathfrak{p}}$.

(d) Follows from part (1).

(4) Proof of First Uniqueness Theorem: Let R be Noetherian and $I = Q_1 \cap \cdots \cap Q_n$ be a minimal primary decomposition.

(a) Use the Lemma to prove that $\mathrm{Ass}_R(R/I) \subseteq \{\sqrt{Q_1}, \ldots, \sqrt{Q_n}\}$.

(b) Set $J_i = \bigcap_{j \neq i} Q_j$. Explain why it suffices to show that $\mathrm{Ass}_R(J_i/I) = \{\sqrt{Q_i}\}$ to establish the other containment.

(c) Let $\mathfrak{q}$ be an associated prime of J_i/I and $r \in R$ such that $\overline{r} \in J_i/I$ is a witness (and in particular, nonzero). Show that $Q_i \subseteq \mathfrak{q}$ and deduce that $\sqrt{Q_i} \subseteq \mathfrak{q}$.

(d) Use the definition of primary to show that $\mathfrak{q} \subseteq \sqrt{Q_i}$, and conclude the proof.

(a) Yes, it is immediate from the lemma.

(b) Because $J_i/I \subseteq R/I$ so $\mathrm{Ass}_R(J_i/I) \subseteq \mathrm{Ass}_R(R/I)$.

[3] One possibility is to consider the support of R/Q.

(c) We have $Q_i r \subseteq Q_i \cap J_i \subseteq I$, so $Q_i \subseteq \operatorname{ann}_R(\overline{r}) = \mathfrak{q}$. Since $\sqrt{Q_i}$ is the unique minimal prime of Q_i and $\mathfrak{q}$ is a prime containing Q_i, we have $\mathfrak{q} \supseteq \sqrt{Q_i}$.

(d) Let $q \in \mathfrak{q}$, so $qr \in I \subseteq Q_i$. Since $\overline{r} \neq 0$, we have $r \notin Q_i$, so by definition of primary, $q \in \sqrt{Q_i}$. Thus $\mathfrak{q} \subseteq \sqrt{Q_i}$. This shows that $\sqrt{Q_i} = \mathfrak{q}$ is an associated prime of J_i/I and hence of R/I.

(5) Prove the Lemma.

(6) Let R be a Noetherian ring, and I be an ideal. Consider a collection of minimal primary decompositions of I:

$$I = \mathfrak{q}_{1,\alpha} \cap \cdots \cap \mathfrak{q}_{s,\alpha}, \quad \alpha \in \Lambda$$

where, for each α, $\sqrt{\mathfrak{q}_{i,\alpha}} = \mathfrak{p}_i$.

(a) Suppose that $\mathfrak{p}_j$ is not contained in any other associated prime of I, and let $W = R \smallsetminus \bigcup_{i \neq j} \mathfrak{p}_i$. Find some minimal primary decompositions of $I(W^{-1}R) \cap R$.

(b) Show (by induction on s) that if we take components $\mathfrak{q}_{1,\alpha_1}, \dots, \mathfrak{q}_{s,\alpha_s}$ from different primary decompositions of I, that we can put them together to get a primary decomposition of I; namely $I = \mathfrak{q}_{1,\alpha_1} \cap \cdots \cap \mathfrak{q}_{s,\alpha_s}$.