DEFINITION: A **minimal primary decomposition** of an ideal I is a primary decomposition

$$I = Q_1 \cap \cdots \cap Q_n$$

such that $Q_i \not\supseteq \bigcap_{j\neq i} Q_j$, and $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$.

THEOREM (FIRST UNIQUENESS THEOREM FOR PRIMARY DECOMPOSITION): Let R be a Noetherian ring and I an ideal. Let

$$I = Q_1 \cap \cdots \cap Q_n$$

be a minimal primary decomposition of I. Then

$$\{\sqrt{Q_1}, \dots, \sqrt{Q_n}\} = \mathrm{Ass}_R(R/I).$$

In particular, the set of primes occurring as the radicals of the primary components are uniquely determined.

THEOREM (SECOND UNIQUENESS THEOREM FOR PRIMARY DECOMPOSITION): Let R be a Noetherian ring and I an ideal. Let

$$I = Q_1 \cap \cdots \cap Q_n$$

be a minimal primary decomposition of I. Suppose that $\mathfrak{p} = \sqrt{Q_i}$ is a *minimal* prime of I. Then $Q_i = IR_{\mathfrak{p}} \cap R$. In particular, the primary components corresponding to the minimal primes are uniquely determined.

LEMMA: Let $I_1, \dots, I_t$ be ideals. Then

(1) for any multiplicatively closed set W, $W^{-1}(I_1 \cap \cdots \cap I_t) = W^{-1}I_1 \cap \cdots \cap W^{-1}I_t$.
(2) $\mathrm{Ass}_R\left(R/\bigcap_{i=1}^t I_i\right) \subseteq \bigcup_{i=1}^t \mathrm{Ass}_R(R/I_i)$.

(1) Uniqueness theorems:
- **(a)** Let K be a field, $R = K[X,Y]$ a polynomial ring, and $I = (X^2, XY)$. Verify[1] that $I = (X) \cap (X^2, Y) = (X) \cap (X^2, XY, Y^2)$ gives two different minimal primary decompositions of I.
- **(b)** In the previous part, which aspects of the decomposition are the same, and which are different. Compare with the uniqueness theorems.
- **(c)** Use the uniqueness theorems to explain why, for $n \in \mathbb{Z}$ with prime factorization $n = \pm p_1^{e_1} \cdots p_m^{e_m}$, the *only*[2] minimal primary decomposition of (n) is

$$(n) = (p_1^{e_1}) \cap \cdots \cap (p_m^{e_m}).$$

(2) Minimal primary decompositions: Let R be a Noetherian ring.
- **(a)** Use the Lemma to explain why a finite intersection of $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary.
- **(b)** Explain how to turn a general $I = Q_1 \cap \cdots \cap Q_m$ primary decomposition into a minimal primary decomposition.

[1] You can take for granted that in each case the intersection is I, but explain why the ideals are primary and the minimality hypotheses hold.

[2] We don't care about the order.

(3) Proof of Second Uniqueness Theorem:

(a) Use the definition of primary to show that if Q is $\mathfrak{p}$-primary, then $QR_{\mathfrak{p}} \cap R = Q$.

(b) Show[3] that if Q is $\mathfrak{q}$-primary and $\mathfrak{q} \not\subseteq \mathfrak{p}$, then $QR_{\mathfrak{p}} = R_{\mathfrak{p}}$.

(c) Let R be Noetherian and $I = Q_1 \cap \cdots \cap Q_n$ be a minimal primary decomposition, and $\mathfrak{p} = \sqrt{Q_i}$ a minimal prime of I. Use the Lemma to show that $IR_{\mathfrak{p}} = Q_i R_{\mathfrak{p}}$.

(d) Complete the proof.

(4) Proof of First Uniqueness Theorem: Let R be Noetherian and $I = Q_1 \cap \cdots \cap Q_n$ be a minimal primary decomposition.

(a) Use the Lemma to prove that $\mathrm{Ass}_R(R/I) \subseteq \{\sqrt{Q_1}, \dots, \sqrt{Q_n}\}$.

(b) Set $J_i = \bigcap_{j \neq i} Q_j$. Explain why it suffices to show that $\mathrm{Ass}_R(J_i/I) = \{\sqrt{Q_i}\}$ to establish the other containment.

(c) Let $\mathfrak{q}$ be an associated prime of J_i/I and $r \in R$ such that $\overline{r} \in J_i/I$ is a witness (and in particular, nonzero). Show that $Q_i \subseteq \mathfrak{q}$ and deduce that $\sqrt{Q_i} \subseteq \mathfrak{q}$.

(d) Use the definition of primary to show that $\mathfrak{q} \subseteq \sqrt{Q_i}$, and conclude the proof.

(5) Prove the Lemma.

(6) Let R be a Noetherian ring, and I be an ideal. Consider a collection of minimal primary decompositions of I:

$$I = \mathfrak{q}_{1,\alpha} \cap \cdots \cap \mathfrak{q}_{s,\alpha}, \quad \alpha \in \Lambda$$

where, for each α, $\sqrt{\mathfrak{q}_{i,\alpha}} = \mathfrak{p}_i$.

(a) Suppose that $\mathfrak{p}_j$ is not contained in any other associated prime of I, and let $W = R \setminus \bigcup_{i \neq j} \mathfrak{p}_i$. Find some minimal primary decompositions of $I(W^{-1}R) \cap R$.

(b) Show (by induction on s) that if we take components $\mathfrak{q}_{1,\alpha_1}, \dots, \mathfrak{q}_{s,\alpha_s}$ from different primary decompositions of I, that we can put them together to get a primary decomposition of I; namely $I = \mathfrak{q}_{1,\alpha_1} \cap \cdots \cap \mathfrak{q}_{s,\alpha_s}$.

[3] One possibility is to consider the support of R/Q.