DEFINITION: A proper ideal I is **primary** if $rs \in I$ implies $r \in \sqrt{I}$ or $s \in I$. We say that I is **$\mathfrak{p}$-primary** if it is primary and $\sqrt{I} = \mathfrak{p}$.

LEMMA: Let R be a Noetherian ring and I an ideal. The following are equivalent:

(i) I is primary;
(ii) Every zerodivisor on R/I is nilpotent;
(iii) $\mathrm{Ass}_R(R/I)$ is a singleton.

DEFINITION: A **primary decomposition** of an ideal I is an expression of the form

$$I = Q_1 \cap \cdots Q_n$$

where each Q_i is a primary ideal.

DEFINITION: A proper ideal I is **irreducible** if $I = J_1 \cap J_2$ for some ideals J_1, J_2 implies $I = J_1$ or $I = J_2$.

THEOREM (EXISTENCE OF PRIMARY DECOMPOSITION): Let R be a Noetherian ring.

(1) Every irreducible ideal I is primary.
(2) Every ideal can be written as a finite intersection of irreducible ideals.

Hence, every ideal can be written as a finite intersection of primary ideals.

(1) Primary ideals

- **(a)** Use the definition to show that a prime ideal is primary.
- **(b)** Use the definition to show that the radical of a primary ideal is prime.
- **(c)** Use the definition to show that for the ideal $I = (X^2, XY)$ in $R = \mathbb{Q}[X, Y]$, $\sqrt{I}$ is prime but I is not primary.
- **(d)** Use the definition and part (b) above to show that if R is a UFD, then a proper principal ideal (f) is primary if and only if it is not generated[1] by a power of a prime element.
- **(e)** Use the Lemma to show that if $\sqrt{I} = \mathfrak{m}$ is a maximal ideal, then I is $\mathfrak{m}$-primary.

(a) A prime ideal is radical in particular, so if Q is prime and $rs \in Q$ and $r \notin \sqrt{Q} = Q$, then $s \in Q$.

(b) Let Q be primary. Suppose that $rs \in \sqrt{Q}$. Then for some n, $r^n s^n = (rs)^n \in Q$ so either $r^n \in \sqrt{Q}$ (whence $r \in \sqrt{Q}$) or $s^n \in Q$ (whence $s \in \sqrt{Q}$).

(c) We have computed $\sqrt{I} = (X)$ earlier, so $\sqrt{I}$ is prime. This ideal is not primary since $XY \in I$ but $X \notin I$ and $Y \notin \sqrt{I}$.

(d) Suppose that $(f) = (r^n)$ for some irreducible r. If $xy \in (f)$, then $r^n | (xy)$, so either $r | x$ (whence $x \in \sqrt{(f)}$) or $r^n | y$ (whence $y \in (f)$). Conversely, suppose that f admits a factorization $f = gh$ with g, h coprime. Then $gh \in (f)$, but $g \notin \sqrt{(f)}$ and $h \notin (f)$.

[1]Note that if (f) is not generated by a power of a prime element, then f has nonassociate irreducible factors.

(e) If $\sqrt{I} = \mathfrak{m}$, then $V(I) = \{\mathfrak{m}\}$ and since $\varnothing \neq \mathrm{Ass}_R(R/I) \subseteq V(I)$, we must have $\mathrm{Ass}_R(R/I) = \{\mathfrak{m}\}$.

(2) Primary decompositions

(a) Let n be an integer. Show that if $n = \pm p_1^{e_1} \cdots p_m^{e_m}$ is the prime factorization of n, then

$$(n) = (p_1^{e_1}) \cap \cdots \cap (p_m^{e_m})$$

is a primary decomposition of (n) in $\mathbb{Z}$.

(b) Let R be a Noetherian ring and I be a radical ideal. Give a recipe for a primary decomposition of I in terms of other named things pertaining to I.

(a) The equality is clear, and each $(p_i^{m_i})$ is primary by above.

(b) $I = \bigcap_{\mathfrak{p} \in \mathrm{Min}(I)} \mathfrak{p}$.

(3) Prove[2] the Lemma.

The equivalence between (i) and (ii) is straightforward. For the (ii)⇔(iii), recall that the set elements of R that are zerodivisors modulo I is the union of the associated primes of R/I and the set of elements that are nilpotent modulo I is the intersection of minimal primes of I. Every minimal prime of I is associated. Thus, if every zerodivisor is nilpotent, then there must be one associated prime (because the union of two distinct sets is always larger than the intersection. Conversely, if there is only one associated prime, the union is the intersection and (ii) holds.

(4) Proof of Existence of Primary Decompositions:

(a) Prove[3] part (2) of the Theorem.

(b) Suppose that $xy \in Q$ with $x \notin Q$ and $y \notin \sqrt{Q}$. Explain why the there is some $n \geq 1$ such that $(Q : y^n) = (Q : y^{n+1})$.

(c) Show that $Q = (Q, x) \cap (Q, y^n)$ and deduce part (1) of the Theorem.

(a) Consider the collection of ideals that are not finite intersections of irreducible ideals. If one exists, by Noetherianity, there is a maximal element I. Such I is necessarily reducible, so $I = J_1 \cap J_2$, with $J_1, J_2 \supsetneq I$. By maximality, J_1, J_2 are finite intersections of irreducible ideals. Substituting in those expressions gives an expression for I as a finite intersection of irreducible ideals.

(b) For each n, we have $(Q : y^n) \subseteq (Q : y^{n+1})$ since $fy^n \in Q$ implies $fy^{n+1} = yfy^n \in Q$. Thus, these ideals form an ascending chain, which must stabilize.

(c) Clearly $Q \subseteq (Q, x) \cap (Q, y^n)$. Write $f = q + ax = q' + by^n$ with $q, q' \in Q$. Then $yf = qy + axy \in Q$, and $yf = q'y + by^{n+1}$, so $by^{n+1} \in Q$. Thus $b \in (Q : y^{n+1}) = (Q : y^n)$, so $by^n \in Q$, but then $f \in Q$. We have shown that if Q is not primary, then it is reducible.

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[2]Hint: For (ii)⇔(iii), recall that the set elements of R that are zerodivisors modulo I is the union of the associated primes of R/I and the set of elements that are nilpotent modulo I is the intersection of minimal primes of I.

[3]Imitate the proof of finiteness of minimal primes.

(5) More examples: Let K be a field.
   (a) Show that $(X^2, XY, Y^2) \subseteq K[X, Y]$ is primary but not irreducible.
   (b) Show that (X^2, XY, Y^3) is primary, but not a power of a prime.
   (c) Show that $(X^2, XY)^2 \subseteq K[X^2, XY, Y^2]$ is a power of a prime but not primary.

> (a) The radical of (X^2, XY, Y^2) is (X, Y), which is maximal, so this is primary. However, $(X^2, XY, Y^2) = (X^2, Y) \cap (X, Y^2)$.
> (b) As above, the radical is (X, Y). Thus, if it is a power of a prime, that must be (X, Y), since the radical of a power of an ideal agree with the radical of the same ideal. Note that $(X, Y)^2 = (X^2, XY, Y^2) \supsetneq (X^2, XY, Y^3) \supsetneq (X, Y)^3$, so this cannot be a power of (X, Y).
> (c) Show that $(X^2, XY)^2 \subseteq K[X^2, XY, Y^2]$ is a power of a prime but not primary.

(6) Let R be a Noetherian ring and $\mathfrak{p}$ a prime ideal. Show that there is an order-preserving bijection

$$\{\mathfrak{p}\text{-primary ideals of } R\} \leftrightarrow \{\text{ideals of } (R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}}) \text{ with radical } \mathfrak{p}R_{\mathfrak{p}}\}.$$

(7) Let R be a Noetherian ring. Show that I is irreducible if and only if it is primary (with radical $\mathfrak{p}$) and $\dfrac{IR_{\mathfrak{p}} : \mathfrak{p}R_{\mathfrak{p}}}{IR_{\mathfrak{p}}}$ is a one-dimensional $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$-vectorspace.