DEFINITION: A proper ideal I is **primary** if $rs \in I$ implies $r \in \sqrt{I}$ or $s \in I$. We say that I is **$\mathfrak{p}$-primary** if it is primary and $\sqrt{I} = \mathfrak{p}$.

LEMMA: Let R be a Noetherian ring and I an ideal. The following are equivalent:

(i) I is primary;
(ii) Every zerodivisor on R/I is nilpotent;
(iii) $\mathrm{Ass}_R(R/I)$ is a singleton.

DEFINITION: A **primary decomposition** of an ideal I is an expression of the form

$$I = Q_1 \cap \cdots Q_n$$

where each Q_i is a primary ideal.

DEFINITION: A proper ideal I is **irreducible** if $I = J_1 \cap J_2$ for some ideals J_1, J_2 implies $I = J_1$ or $I = J_2$.

THEOREM (EXISTENCE OF PRIMARY DECOMPOSITION): Let R be a Noetherian ring.

(1) Every irreducible ideal I is primary.
(2) Every ideal can be written as a finite intersection of irreducible ideals.

Hence, every ideal can be written as a finite intersection of primary ideals.

(1) Primary ideals
- **(a)** Use the definition to show that a prime ideal is primary.
- **(b)** Use the definition to show that the radical of a primary ideal is prime.
- **(c)** Use the definition to show that for the ideal $I = (X^2, XY)$ in $R = \mathbb{Q}[X, Y]$, $\sqrt{I}$ is prime but I is not primary.
- **(d)** Use the definition and part (b) above to show that if R is a UFD, then a proper principal ideal (f) is primary if and only if it is not generated[1] by a power of a prime element.
- **(e)** Use the Lemma to show that if $\sqrt{I} = \mathfrak{m}$ is a maximal ideal, then I is $\mathfrak{m}$-primary.

(2) Primary decompositions
- **(a)** Let n be an integer. Show that if $n = \pm p_1^{e_1} \cdots p_m^{e_m}$ is the prime factorization of n, then

$$(n) = (p_1^{e_1}) \cap \cdots \cap (p_m^{e_m})$$

  is a primary decomposition of (n) in $\mathbb{Z}$.
- **(b)** Let R be a Noetherian ring and I be a radical ideal. Give a recipe for a primary decomposition of I in terms of other named things pertaining to I.

(3) Prove[2] the Lemma.

[1] Note that if (f) is not generated by a power of a prime element, then f has nonassociate irreducible factors.

[2] Hint: For (ii)⇔(iii), recall that the set elements of R that are zerodivisors modulo I is the union of the associated primes of R/I and the set of elements that are nilpotent modulo I is the intersection of minimal primes of I.

(4) Proof of Existence of Primary Decompositions:

(a) Prove[3] part (2) of the Theorem.

(b) Suppose that $xy \in Q$ with $x \notin Q$ and $y \notin \sqrt{Q}$. Explain why the there is some $n \geq 1$ such that $(Q : y^n) = (Q : y^{n+1})$.

(c) Show that $Q = (Q, x) \cap (Q, y^n)$ and deduce part (1) of the Theorem.

(5) More examples: Let K be a field.

(a) Show that $(X^2, XY, Y^2) \subseteq K[X, Y]$ is primary but not irreducible.

(b) Show that (X^2, XY, Y^3) is primary, but not a power of a prime.

(c) Show that $(X^2, XY)^2 \subseteq K[X^2, XY, Y^2]$ is a power of a prime but not primary.

(6) Let R be a Noetherian ring and $\mathfrak{p}$ a prime ideal. Show that there is an order-preserving bijection

$$\{\mathfrak{p}\text{-primary ideals of } R\} \leftrightarrow \{\text{ideals of } (R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p}) \text{ with radical } \mathfrak{p}R_\mathfrak{p}\}.$$

(7) Let R be a Noetherian ring. Show that I is irreducible if and only if it is primary (with radical $\mathfrak{p}$) and $\dfrac{IR_\mathfrak{p} : \mathfrak{p}R_\mathfrak{p}}{IR_\mathfrak{p}}$ is a one-dimensional $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$-vectorspace.

[3] Imitate the proof of finiteness of minimal primes.