LEMMA: Let R be a ring, and $N \subseteq M$ be modules. Then

$$\operatorname{Ass}_R(N) \subseteq \operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(N) \cup \operatorname{Ass}_R(M/N).$$

EXISTENCE OF PRIME FILTRATIONS: Let R be a Noetherian ring and M be a finitely generated module. Then there exists a finite chain of submodules

$$M = M_t \supsetneq M_{t-1} \supsetneq \cdots \supsetneq M_1 \supsetneq M_0 = 0$$

such that for each $i = 1, \dots, t$, there is some $\mathfrak{p}_i \in \operatorname{Spec}(R)$ such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$. Such a chain of submodules is called a **prime filtration** of M.

COROLLARY 1: Let R be a Noetherian ring and M be a finitely generated module. Then for any prime filtration of M, $\operatorname{Ass}_R(M)$ is a subset of the prime factors that occur in the filtration. In particular, $\operatorname{Ass}_R(M)$ is finite.

PRIME AVOIDANCE: Let R be a ring, J an ideal, and $I_1, I_2, I_3, \dots, I_t$ a finite collection of ideals with I_i prime for $i > 2$ (that is, *at most* two I_i are not prime). If $J \not\subseteq I_i$ for all i, then $J \not\subseteq \bigcup_i I_i$.

COROLLARY 2: Let R be a Noetherian ring, M a finitely generated module, and I an ideal. If every element of I is a zerodivisor on M, then there is some nonzero $m \in M$ such that $Im = 0$.

(1) Let $R = K[X,Y]$ and $M = R/(X^2Y, XY^2)$.
- **(a)** Verify that $0 \subseteq Rxy \subseteq Rx \subseteq M$ is a prime filtration of M.
- **(b)** In an earlier problem, we more or less showed that $\{(x), (y), (x,y)\} \subseteq \operatorname{Ass}_R(M)$. Use Corollary 1 to deduce that this is an equality.

- **(a)** We have $Rxy \cong (XY)/(X^2Y, XY^2)$. The elements that multiply XY into (X^2Y, XY^2) are the elements in (X,Y), so this is isomorphic to $R/(X,Y)$ and (X,Y) is prime. Then $Rx/Rxy \cong (X)/(XY, X^2Y, XY^2) = (X)/(XY) \cong R/(Y)$ and Y is prime. Finally, the last quotient is isomorphic to $R/(X)$, and (X) is prime.
- **(b)** Yes, it gives the other containment!

(2) Proving some Corollaries:
- **(a)** Show that Corollary 1 follows from the Lemma (and Existence of Prime Filtrations).
- **(b)** Write the contrapositive of the conclusion of Prime Avoidance.
- **(c)** Show that Corollary 2 follows from Prime Avoidance and Corollary 1.

- **(a)** We just need to show the first statement. By the Lemma, we have $\operatorname{Ass}_R(M) = \operatorname{Ass}_R(M_t) \subseteq \operatorname{Ass}_R(M_{t-1}) \cup \operatorname{Ass}_R(M_t/M_{t-1}) = \operatorname{Ass}_R(M_{t-1}) \cup \{\mathfrak{p}_t\}$. Then $\operatorname{Ass}_R(M_{t-1}) \subseteq \operatorname{Ass}_R(M_{t-2}) \cup \{p_{t-1}\}$. Continuing like so we obtain the conclusion.

(b) If $J \subseteq \bigcup_i I_i$ then $J \subseteq I_i$ for some i.
(c) From last time, we know that the set of zerodivisors is $\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}$. If I contained in this *finite* union of primes, it is contained in one of them by Prime avoidance. But if I is contained in an associated prime, take a witness m, and $Im = 0$.

(3) Proof of Existence of Prime Filtrations: Let R be a Noetherian ring and M a finitely generated R-module.

- **(a)** If $M \neq 0$, explain why you can always choose $M \supseteq M_1$ with $M_1 \cong R/\mathfrak{p}$ for some prime $\mathfrak{p}$.
- **(b)** If $M \neq M_1$, explain why[1] you can always choose $M \supseteq M_2 \supseteq M_1$ with $M_2/M_1 \cong R/\mathfrak{p}$ for some prime $\mathfrak{p}$.
- **(c)** If $M \neq M_{i-1}$ and you already have $M_1, \ldots, M_{i-1}$, explain why you can always choose $M \supseteq M_i \supsetneq M_{i-1}$ with $M_i/M_{i-1} \cong R/\mathfrak{p}$ for some prime $\mathfrak{p}$.
- **(d)** Explain why this process has to stop, and if it stops at $i = t$, we must have $M_t = M$.

(a) M has an associated prime, since R is Noetherian and $M \neq 0$. An associated prime is a recipe for exactly such a submodule.
(b) Apply the previous to M/M_1. By the lattice theorem, we can write this as M_2/M_1 for some M_2 containing M_1.
(c) Same thing.
(d) M is a Noetherian module, so an ascending chain of submodules terminates. It must terminate with $M_t = M$ by what we just said in the previous step.

(4) Lemma 1:

- **(a)** Let K be a field and $R = K[X]$. Explain why
  - $\mathrm{Ass}_R(R) = \{(0)\}$
  - $(X) \cong R$, so $\mathrm{Ass}_R((X)) = \{(0)\}$,
  - $\mathrm{Ass}_R(R/(X)) = \{(X)\}$.

  Does this contradict the Lemma?
- **(b)** Show that $\mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(M)$.
- **(c)** Suppose that $\mathfrak{p} \in \mathrm{Ass}_R(M) \smallsetminus \mathrm{Ass}_R(N)$ with witness m. Show[2] that $Rm \cap N = 0$, so the map $Rm \to M/N$ is injective. Deduce that $\mathfrak{p} \in \mathrm{Ass}_R(M/N)$ and complete the proof.

(a)
- This is an example of $\mathrm{Ass}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$.
- The map $R \to (X)$ given by $r \mapsto rX$ is R-linear and bijective, so an isomorphism of R-modules.
- This is an example of $\mathrm{Ass}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$.

This does not contradict the lemma.
(b) A witness of $\mathfrak{p}$ in N is a witness for $\mathfrak{p}$ in M.

[1] Hint: Consider M/M_1 and go back to the previous step.
[2] Note that $Rm \cong R/\mathfrak{p}$ so every nonzero element has annihilator $\mathfrak{p}$.

(c) Note that $Rm \cong R/\mathfrak{p}$ so every nonzero element has annihilator $\mathfrak{p}$. Since $\mathfrak{p} \notin \operatorname{Ass}_R(N)$, no element of N has annihilator $\mathfrak{p}$, so nonzero element of Rm is also an element of N. Thus the induced map $R/\mathfrak{p} \cong Rm \hookrightarrow M/N$, so $\mathfrak{p} \in \operatorname{Ass}_R(M/N)$.

(5) Prove[3] the prime avoidance lemma.

(6) Let K be a field and $R = K[X^2, XY, Y^2] \subseteq K[X, Y]$.
  (a) Mark all[4] of the points in the plane corresponding to exponent vectors of elements of R.
  (b) Is $I = (X^2)$ a prime ideal? Is $J = (X^2, XY)$?
  (c) Mark all of the points in the plane corresponding to exponent vectors of elements of $(X^2) \subseteq R$.
  (d) Find and illustrate a prime filtration of R/I. Compute $\operatorname{Ass}_R(R/I)$.
  (e) Find and illustrate a prime filtration of R/J^2. Compute $\operatorname{Ass}_R(R/J^2)$.

(7) More facts about associated primes: Let R be a Noetherian ring.
  (a) Let $I \subseteq J$ be ideals. Show that $I = J$ if and only if $IR_{\mathfrak{p}} = JR_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(R/I)$.
  (b) Let I, J be ideals. Show that $I \subseteq J$ if and only if $IR_{\mathfrak{p}} \subseteq JR_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(R/J)$.
  (c) Let r be a nonzerodivisor. Show that $\operatorname{Ass}_R(R/r^n) = \operatorname{Ass}_R(R/r)$ for all $n \geq 1$.

[3] By induction, you can find elements $a_i \in J \smallsetminus \bigcup_{j \neq i} I_j$. Now consider $x = a_n + a_1 \cdots a_{n-1}$.

[4] Well, enough to get the pattern at least…