LEMMA: Let R be a ring, and $N \subseteq M$ be modules. Then
$$\operatorname{Ass}_R(N) \subseteq \operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(N) \cup \operatorname{Ass}_R(M/N).$$

EXISTENCE OF PRIME FILTRATIONS: Let R be a Noetherian ring and M be a finitely generated module. Then there exists a finite chain of submodules
$$M = M_t \supsetneq M_{t-1} \supsetneq \cdots \supsetneq M_1 \supsetneq M_0 = 0$$
such that for each $i = 1, \ldots, t$, there is some $\mathfrak{p}_i \in \operatorname{Spec}(R)$ such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$. Such a chain of submodules is called a **prime filtration** of M.

COROLLARY 1: Let R be a Noetherian ring and M be a finitely generated module. Then for any prime filtration of M, $\operatorname{Ass}_R(M)$ is a subset of the prime factors that occur in the filtration. In particular, $\operatorname{Ass}_R(M)$ is finite.

PRIME AVOIDANCE: Let R be a ring, J an ideal, and $I_1, I_2, I_3, \ldots, I_t$ a finite collection of ideals with I_i prime for $i > 2$ (that is, *at most* two I_i are not prime). If $J \not\subseteq I_i$ for all i, then $J \not\subseteq \bigcup_i I_i$.

COROLLARY 2: Let R be a Noetherian ring, M a finitely generated module, and I an ideal. If every element of I is a zerodivisor on M, then there is some nonzero $m \in M$ such that $Im = 0$.

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(1) Let $R = K[X, Y]$ and $M = R/(X^2Y, XY^2)$.
  (a) Verify that $0 \subseteq Rxy \subseteq Rx \subseteq M$ is a prime filtration of M.
  (b) In an earlier problem, we more or less showed that $\{(x), (y), (x, y)\} \subseteq \operatorname{Ass}_R(M)$. Use Corollary 1 to deduce that this is an equality.

(2) Proving some Corollaries:
  (a) Show that Corollary 1 follows from the Lemma (and Existence of Prime Filtrations).
  (b) Write the contrapositive of the conclusion of Prime Avoidance.
  (c) Show that Corollary 2 follows from Prime Avoidance and Corollary 1.

(3) Proof of Existence of Prime Filtrations: Let R be a Noetherian ring and M a finitely generated R-module.
  (a) If $M \neq 0$, explain why you can always choose $M \supseteq M_1$ with $M_1 \cong R/\mathfrak{p}$ for some prime $\mathfrak{p}$.
  (b) If $M \neq M_1$, explain why[1] you can always choose $M \supseteq M_2 \supseteq M_1$ with $M_2/M_1 \cong R/\mathfrak{p}$ for some prime $\mathfrak{p}$.
  (c) If $M \neq M_{i-1}$ and you already have $M_1, \ldots, M_{i-1}$, explain why you can always choose $M \supseteq M_i \supsetneq M_{i-1}$ with $M_i/M_{i-1} \cong R/\mathfrak{p}$ for some prime $\mathfrak{p}$.
  (d) Explain why this process has to stop, and if it stops at $i = t$, we must have $M_t = M$.

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[1]Hint: Consider M/M_1 and go back to the previous step.

(4) Lemma 1:

    (a) Let K be a field and $R = K[X]$. Explain why
- $\mathrm{Ass}_R(R) = \{(0)\}$
- $(X) \cong R$, so $\mathrm{Ass}_R((X)) = \{(0)\}$,
- $\mathrm{Ass}_R(R/(X)) = \{(X)\}$.

    Does this contradict the Lemma?

    (b) Show that $\mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(M)$.

    (c) Suppose that $\mathfrak{p} \in \mathrm{Ass}_R(M) \smallsetminus \mathrm{Ass}_R(N)$ with witness m. Show[2] that $Rm \cap N = 0$, so the map $Rm \to M/N$ is injective. Deduce that $\mathfrak{p} \in \mathrm{Ass}_R(M/N)$ and complete the proof.

(5) Prove[3] the prime avoidance lemma.

(6) Let K be a field and $R = K[X^2, XY, Y^2] \subseteq K[X, Y]$.

    (a) Mark all[4] of the points in the plane corresponding to exponent vectors of elements of R.

    (b) Is $I = (X^2)$ a prime ideal? Is $J = (X^2, XY)$?

    (c) Mark all of the points in the plane corresponding to exponent vectors of elements of $(X^2) \subseteq R$.

    (d) Find and illustrate a prime filtration of R/I. Compute $\mathrm{Ass}_R(R/I)$.

    (e) Find and illustrate a prime filtration of R/J^2. Compute $\mathrm{Ass}_R(R/J^2)$.

(7) More facts about associated primes: Let R be a Noetherian ring.

    (a) Let $I \subseteq J$ be ideals. Show that $I = J$ if and only if $IR_{\mathfrak{p}} = JR_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$.

    (b) Let I, J be ideals. Show that $I \subseteq J$ if and only if $IR_{\mathfrak{p}} \subseteq JR_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Ass}_R(R/J)$.

    (c) Let r be a nonzerodivisor. Show that $\mathrm{Ass}_R(R/r^n) = \mathrm{Ass}_R(R/r)$ for all $n \geq 1$.

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[2] Note that $Rm \cong R/\mathfrak{p}$ so every nonzero element has annihilator $\mathfrak{p}$.

[3] By induction, you can find elements $a_i \in J \smallsetminus \bigcup_{j \neq i} I_j$. Now consider $x = a_n + a_1 \cdots a_{n-1}$.

[4] Well, enough to get the pattern at least...