DEFINITION: Let R be a ring and M be a module. A prime ideal $\mathfrak{p}$ of R is an **associated prime** of M if $\mathfrak{p} = \operatorname{ann}_R(m)$ for some $m \in M$. The element m is called a **witness** for the associated prime $\mathfrak{p}$. We write $\operatorname{Ass}_R(M)$ for the set of associated primes of a module.

LEMMA: Let R be a Noetherian ring and M be a module. For any nonzero element $m \in M$, the ideal $\operatorname{ann}_R(m)$ is contained in an associated prime of M. In particular, if $M \neq 0$, then M has an associated prime.

DEFINITION: Let R be a ring and M be an R-module. We say that an element $r \in R$ is a **zerodivisor** on M if there is some $m \in M \smallsetminus 0$ such that $rm = 0$.

PROPOSITION: Let R be a Noetherian ring and M an R-module. The set of zerodivisors on M is the union of the associated primes of M.

THEOREM: Let R be a Noetherian ring, W be a multiplicatively closed set, and M be a module. Then

$$\operatorname{Ass}_{W^{-1}R}(W^{-1}M) = \{W^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R(M), \mathfrak{p} \cap W = \varnothing\}.$$

COROLLARY: Let R be a Noetherian ring and I be an ideal. Then $\operatorname{Min}(I) \subseteq \operatorname{Ass}_R(R/I)$.

(1) Proof of Lemma and Proposition: Let R be a Noetherian ring and M be a nonzero module.
- **(a)** Let $\mathcal{S} = \{\operatorname{ann}_R(m) \mid m \in M \smallsetminus 0\}$. Explain why $\mathcal{S}$ has a maximal element J.
- **(b)** Let $J = \operatorname{ann}_R(m)$ and suppose that $rs \in J$ but $s \notin J$. Explain why $J = \operatorname{ann}_R(sm)$.
- **(c)** Conclude the proof of the Lemma.
- **(d)** Deduce the Proposition from the Lemma.
- **(e)** What does the Proposition say in the special case when $M = R$?

- **(a)** Because this is a nonempty collection of ideals in a Noetherian ring.
- **(b)** First, $\operatorname{ann}_R(sm) \supseteq \operatorname{ann}_R(m)$ since $rm = 0$ implies $rsm = 0$. Since $s \notin J$, $\operatorname{ann}_R(sm) \neq R$, so by maximality we have equality.
- **(c)** Suppose $s \notin J$ and $rs \in J$. Then $rsm = 0$ implies that $r \in \operatorname{ann}_R(sm) = J$. Thus J is prime. Since any element of $\mathcal{S}$ is contained in a maximal element, the claim follows.
- **(d)** If r is a zerodivisor on M, then r is contained in some ideal of $\mathcal{S}$, and then it is contained in an associated prime. Conversely, any element in an associated prime is a zerodivisor on M by definition.
- **(e)** The zerodivisors in R are the elements in some associated prime.

(2) Working with associated primes.
- **(a)** Let R be a domain and M be a torsionfree module. Show that $\operatorname{Ass}_R(M) = \{(0)\}$.
- **(b)** Let R be a ring and $\mathfrak{p}$ be a prime ideal. Show that for any nonzero element $\overline{r} \in R/\mathfrak{p}$ that $\operatorname{ann}_R(\overline{r}) = \mathfrak{p}$ and use the definition to deduce that $\operatorname{Ass}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$.

(c) Let K be a field and $R = K[X,Y]/(X^2Y, XY^2)$. Use[1] the definition to show that (x,y), (x), and (y) are associated primes of R.

(d) Let M be a module. Explain why $\mathfrak{p} \in \operatorname{Ass}_R(M)$ if and only if there is an injective R-module homomorphism $R/\mathfrak{p} \hookrightarrow M$.

> **(a)** By definition, any nonzero element has annihilator zero.
>
> **(b)** Clearly $\mathfrak{p} \subseteq \operatorname{ann}_R(\overline{r})$. Let r be a representative of $\overline{r}$; we have $r \notin \mathfrak{p}$. The annihilator of $\overline{r} \in R/\mathfrak{p}$ is the set of $s \in R$ such that $sr \in \mathfrak{p}$. By definition of prime, $s \in \mathfrak{p}$, so $\operatorname{ann}_R(\overline{r}) \subseteq \mathfrak{p}$ and equality holds.
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> **(c)** Since $x \cdot xy = x^2y = 0$ and $y \cdot xy = xy^2$, the annihilator of xy contains (x,y); any element not in (x,y) has some/every representative with a nonzero constant term, and hence does not kill xy. Thus $\operatorname{ann}_R(xy) = (x,y)$.
> We claim that $\operatorname{ann}_R(y^2) = (x)$. Indeed, $x \cdot y^2 = 0$, and if $f \notin (x)$, then some/every representative f has a nonzero term that only involves Y, and $f \cdot Y^2$ has a noznero term only involving Y, and hence nonzero modulo (X^2Y, XY^2). The claim follows.
> Along similar lines, $\operatorname{ann}_R(x^2) = (y)$.
>
> **(d)** If $\operatorname{ann}_R(m) = \mathfrak{p}$, then the map $R \to M$ sending $1 \mapsto m$ has kernel $\mathfrak{p}$, so one has an injection $R/\mathfrak{p} \to M$. Conversely, if $R/\mathfrak{p} \hookrightarrow M$, then the image of 1 has annihilator $\mathfrak{p}$.

(3) Using the Theorem. Let R be a Noetherian ring.

(a) Restate the Theorem in the special case $W = R \smallsetminus \mathfrak{p}$ with our standard notation for this setting.

(b) Show (either using the Theorem or 2(d) above) that $\operatorname{Ass}_R(M) \subseteq \operatorname{Supp}_R(M)$.

(c) Use the Theorem (and the previous part or otherwise) to prove the Corollary.

(d) Show the more general statement: if M is a nonzero module, then the primes that are minimal within the support of I are associated to M.

> **(a)** $\operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Ass}_R(M) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}$.
>
> **(b)** Suppose that $\mathfrak{p} \in \operatorname{Ass}_R(M)$. Then $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$ so $M_\mathfrak{p} \neq 0$.
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> **(c)** Let $M = R/I$ and $\mathfrak{p} \in \operatorname{Min}(I)$. Then $M_\mathfrak{p} \neq 0$ (for various reasons as previously discussed in localizations), so $M_\mathfrak{p} \neq 0$. But the support of $M_\mathfrak{p}$ is $V(IR_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$, so $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$ and hence $\mathfrak{p} \in \operatorname{Ass}_R(M)$.
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> **(d)** The previous argument shows this.

(4) The ring of Puiseux series is $R = \bigcup_{n \geq 1} \mathbb{C}[\![X^{1/n}]\!]$: elements consist of power series with fractional exponents that have a common denominator (though different elements can have different common denominators).

(a) Show that every nonzero element of R can be written in the form $X^{m/n} \cdot u$ for some unit u.

(b) Show that the R-module $R/(X)$ is nonzero but has no associated primes.

(5) Proof of Theorem: Let R be a Noetherian ring, W be a multiplicatively closed set, and M be a module.

[1] Hint: Consider xy and y^2.

(a) Suppose that $\mathfrak{p}$ is an associated prime of M with $W \cap \mathfrak{p} = \varnothing$, and let m be a witness for $\mathfrak{p}$ as an associated prime of M. Show that $W^{-1}\mathfrak{p}$ is an associated prime of $W^{-1}M$ with witness $\frac{m}{1}$.
(b) Suppose that $W^{-1}\mathfrak{p} \in \operatorname{Spec}(W^{-1}R)$ is an associated prime of $W^{-1}M$. Explain why there is a witness of the form $\frac{m}{1}$.
(c) Let $\mathfrak{p} = (f_1, \dots, f_t)$. Explain why there exist $w_1, \dots, w_t \in W$ such that $w_i f_i m = 0$ in M for all i.
(d) Show that $w_1 \cdots w_t m$ is a witness for $\mathfrak{p}$ as an associated prime of M.

(6) Let R be a Noetherian ring and M be a module. Show that $\mathfrak{p} \in \operatorname{Ass}_R(M)$ if and only if for every $r \in \mathfrak{p}$ and every nonzero $m \in M$, there exists some $u \notin \mathfrak{p}$ such that $urm = 0$.

(7) Let R be a Noetherian ring. Is every minimal prime of a zerodivisor a minimal prime of R?