THEOREM: Let R be a Noetherian ring. Every ideal of R has finitely many minimal primes.

LEMMA: Let R be a ring, I an ideal, and $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ a finite set of incomparable prime ideals; i.e., $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for any $i \neq j$. If $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$, then $\mathrm{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$.

COROLLARY: Let R be a Noetherian ring. Every radical ideal of R can be written as a finite intersection of primes in a unique way such that no term can be omitted.

(1) Minimal primes review:
- **(a)** What is the intersection of all minimal primes of R?
- **(b)** What is the intersection of all minimal primes of I?
- **(c)** Explain why an arbitrary intersection of prime ideals is radical.
- **(d)** Explain why any radical ideal is an intersection of prime ideals.

> **(a)** The nilradical: set of nilpotents.
> **(b)** The radical of I.
> **(c)** Follows from the definition of prime.
> **(d)** Formal Nullstellensatz.

(2) Proof of Theorem: Let R be a Noetherian ring.
- **(a)** Suppose the conclusion is false. Explain why[1] the set of ideals that do not have finitely many minimal primes has a maximal element J.
- **(b)** Explain why J is not prime.
- **(c)** Explain why, if $ab \in J$, $V(J) = V(J + (a)) \cup V(J + (b))$; i.e., every prime that contains J either contains $J + (a)$ or $J + (b)$.
- **(d)** Conclude the proof.

> **(a)** In a Noetherian ring, any nonempty family of ideals has a maximal element.
> **(b)** A prime has one minimal prime.
> **(c)** If $\mathfrak{p} \supseteq J \ni ab$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. In the first case, $J + (a) \subseteq \mathfrak{p}$; similarly in the second.
> **(d)** By minimality, $J + (a)$ and $J + (b)$ have finitely many minimal primes. But any minimal prime of J is a minimal prime of $J + (a)$ or $J + (b)$, and the union of these sets is finite, so J has finitely many minimal primes.

(3) In this problem, we will show that the minimal primes of $R = \mathbb{Q}[X, Y, Z, W]/(X^2 - Z^2, XY - ZW, Y^2 - W^2)$ are $(x-z, y-w)$ and $(x+z, y+w)$. Equivalently, we show that the minimal primes of $I = (X^2 - Z^2, XY - ZW, Y^2 - W^2)$ are $(X + Z, Y - W)$ and $(X + Z, Y + W)$.
- **(a)** Factor the first and last relations to show that any prime containing I contains either $X - Z$ or $X + Z$, and also contains either $Y - W$ or $Y + W$.

[1] Warning: this looks like cause to apply Zorn's Lemma, but that is not why.

(b) Show[2] that $(X-Z, Y-W) \supseteq I$ and $(X+Z, Y+W) \supseteq I$.

(c) Show that $XY \in (X-Z, Y+W)+I$. Deduce that any prime that contains $(X-Z, Y+W)$ and I also contains either $(X-Z, Y-W)$ or $(X+Z, Y+W)$.

(d) Deduce the claim.

> **(a)** If $\mathfrak{p} \supseteq I$, then $\mathfrak{p} \ni (X-Z)(X+W)$, so $\mathfrak{p} \ni X-Z$ or $\mathfrak{p} \ni X+Z$. Likewise with $Y \pm W$.
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> **(b)** We have $X^2-Z^2, Y^2-W^2 \in (X-Z, Y-W)$, so we just need to check that $XY-ZW \in (X-Z, Y-W)$. We have $XY-ZW \equiv XY-XY \equiv 0 \mod (X-Z, Y-W)$. Similarly for the other.
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> **(c)** We have $XY-ZW \equiv XY-X(-Y) = 2XY \mod (X-Z, Y+W)$; dividing by 2, we get $XY \in (X-Z, Y+W)+I$. Then any prime containing $(X-Z, Y+W)$ and I contains $X-Z, Y+W$ and either X or Y, but given X, the prime contains $(X, Z, Y+W) \supseteq (X+Z, Y+W)$. Similarly if $\mathfrak{p}$ contains $X-Z, Y+W, Y$, then $\mathfrak{p}$ contains $(X-Z, Y, W) \supseteq (X-Z, Y-W)$.
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> **(d)** One deduces similarly to above that a prime containing I and $(X+Z, Y-W)$ contains one of the given primes. Thus any prime containing I contains $(X-Z, Y-W)$ or $(X+Z, Y+W)$, so these are the minimal primes.

(4) (a) Use the Theorem to show that, if R is Noetherian, a subset of $\mathrm{Spec}(R)$ is closed if and only if it is a finite union of "upward intervals" $V(\mathfrak{p}_i)$.
(b) Use the Theorem to show that, if R is Noetherian, then $\mathrm{Min}(R)$ is discrete.
(c) Prove the Lemma.
(d) Prove the Corollary.

> (a) Follows from the fact that every $p \in V(I)$ contains a minimal prime of I.
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> (b) Every point is closed, and the set is finite, so any subset is closed.
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> (c) Suppose that $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$ with $\mathfrak{p}_i$ incomparable. Note that each $\mathfrak{p}_i$ contains I. Suppose that $\mathfrak{q} \supseteq I$. We claim that $\mathfrak{q}$ contains some $\mathfrak{p}_i$; if not, take $f_i \in \mathfrak{p}_i \setminus \mathfrak{q}$; then $f_1 \cdots f_t \in I \setminus \mathfrak{q}$, a contradiction. It follows that any minimal prime is some $\mathfrak{p}_i$, and each is minimal by incomparability.
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> (d) Every radical ideal is the intersection of its minimal primes.

(5) (a) Compute the minimal primes of $R = \mathbb{Q}[X,Y,Z]/(XY, XZ, YZ)$.
(b) Compute the minimal primes of $R = \mathbb{Q}[X,Y,Z]/(X^2-X^3, XY^3, XZ^4-Z^4)$.

(6) Let K be a field. Let $R = \dfrac{K[X_1, X_2, X_3, \dots, Y_1, Y_2, Y_3, \dots]}{(\{X_iY_i \mid i \geq 1\})}$. Compute $\mathrm{Min}(R)$, and show that $(x_1, x_2, x_3, \dots)$ is not open in $\mathrm{Min}(R)$; in particular, $\mathrm{Min}(R)$ is not discrete.

(7) Let K be a field. Let $R = \dfrac{K[X_1, X_2, X_3, \dots]}{(\{X_iX_j - X_j \mid 1 \leq i \leq j\})}$. Compute $\mathrm{Min}(R)$, and show that $(x_1, x_2, x_3, \dots)$ is not open in $\mathrm{Min}(R)$; in particular, $\mathrm{Min}(R)$ is not discrete.

[2] Hint: Sometimes if you want to show $f \in J$ it is cleanest to show $f \equiv 0 \mod J$.