THEOREM: Let R be a Noetherian ring. Every ideal of R has finitely many minimal primes.

LEMMA: Let R be a ring, I an ideal, and $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ a finite set of incomparable prime ideals; i.e., $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for any $i \neq j$. If $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$, then $\operatorname{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$.

COROLLARY: Let R be a Noetherian ring. Every radical ideal of R can be written as a finite intersection of primes in a unique way such that no term can be omitted.

(1) Minimal primes review:
- **(a)** What is the intersection of all minimal primes of R?
- **(b)** What is the intersection of all minimal primes of I?
- **(c)** Explain why an arbitrary intersection of prime ideals is radical.
- **(d)** Explain why any radical ideal is an intersection of prime ideals.

(2) Proof of Theorem: Let R be a Noetherian ring.
- **(a)** Suppose the conclusion is false. Explain why[1] the set of ideals that do not have finitely many minimal primes has a maximal element J.
- **(b)** Explain why J is not prime.
- **(c)** Explain why, if $ab \in J$, $V(J) = V(J + (a)) \cup V(J + (b))$; i.e., every prime that contains J either contains $J + (a)$ or $J + (b)$.
- **(d)** Conclude the proof.

(3) In this problem, we will show that the minimal primes of $R = \mathbb{Q}[X, Y, Z, W]/(X^2 - Z^2, XY - ZW, Y^2 - W^2)$ are $(x-z, y-w)$ and $(x+z, y+w)$. Equivalently, we show that the minimal primes of $I = (X^2 - Z^2, XY - ZW, Y^2 - W^2)$ are $(X + Z, Y - W)$ and $(X + Z, Y + W)$.
- **(a)** Factor the first and last relations to show that any prime containing I contains either $X - Z$ or $X + Z$, and also contains either $Y - W$ or $Y + W$.
- **(b)** Show[2] that $(X - Z, Y - W) \supseteq I$ and $(X + Z, Y + W) \supseteq I$.
- **(c)** Show that $XY \in (X - Z, Y + W) + I$. Deduce that any prime that contains $(X - Z, Y + W)$ and I also contains either $(X - Z, Y - W)$ or $(X + Z, Y + W)$.
- **(d)** Deduce the claim.

(4)
- (a) Use the Theorem to show that, if R is Noetherian, a subset of $\operatorname{Spec}(R)$ is closed if and only if it is a finite union of "upward intervals" $V(\mathfrak{p}_i)$.
- (b) Use the Theorem to show that, if R is Noetherian, then $\operatorname{Min}(R)$ is discrete.
- (c) Prove the Lemma.
- (d) Prove the Corollary.

(5)
- (a) Compute the minimal primes of $R = \mathbb{Q}[X, Y, Z]/(XY, XZ, YZ)$.
- (b) Compute the minimal primes of $R = \mathbb{Q}[X, Y, Z]/(X^2 - X^3, XY^3, XZ^4 - Z^4)$.

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[1] Warning: this looks like cause to apply Zorn's Lemma, but that is not why.

[2] Hint: Sometimes if you want to show $f \in J$ it is cleanest to show $f \equiv 0 \mod J$.

(6) Let K be a field. Let $R = \dfrac{K[X_1, X_2, X_3, \dots, Y_1, Y_2, Y_3, \dots]}{(\{X_i Y_i \mid i \geq 1\})}$. Compute $\operatorname{Min}(R)$, and show that $(x_1, x_2, x_3, \dots)$ is not open in $\operatorname{Min}(R)$; in particular, $\operatorname{Min}(R)$ is not discrete.

(7) Let K be a field. Let $R = \dfrac{K[X_1, X_2, X_3, \dots]}{(\{X_i X_j - X_j \mid 1 \leq i \leq j\})}$. Compute $\operatorname{Min}(R)$, and show that $(x_1, x_2, x_3, \dots)$ is not open in $\operatorname{Min}(R)$; in particular, $\operatorname{Min}(R)$ is not discrete.