DEFINITION: Let $\mathcal{P}$ be a property[1] of a ring. We say that

- $\mathcal{P}$ is **preserved by localization** if
$$\mathcal{P} \text{ holds for } R \Longrightarrow \text{for every multiplicatively closed set } W, \mathcal{P} \text{ holds for } W^{-1}R.$$
- $\mathcal{P}$ is a **local property** if
$$\mathcal{P} \text{ holds for } R \Longleftrightarrow \text{for every prime ideal } \mathfrak{p} \in \operatorname{Spec}(R), \mathcal{P} \text{ holds for } R_{\mathfrak{p}}.$$

One defines **preserved by localization** and **local property** for properties of modules in the same way, or for properties of a ring element (where one considers $\frac{r}{1} \in W^{-1}R$ or $R_{\mathfrak{p}}$ in the right-hand side) or module element.

DEFINITION: The **support** of a module M is
$$\{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}.$$

PROPOSITION: If M is a finitely generated module, then $\operatorname{Supp}(M) = V(\operatorname{ann}_R(M))$.

(1) Let R be a ring, M be a module, and $m \in M$.

(a) Show that[2] the following are equivalent:

(i) $m = 0$ in M;
(ii) $\frac{m}{1} = 0$ in $W^{-1}M$ for all multiplicatively closed $W \subseteq R$;
(iii) $\frac{m}{1} = 0$ in $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$;
(iv) $\frac{m}{1} = 0$ in $M_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Max}(R)$.

(b) Deduce that "$= 0$" (as a property of a module element) is preserved by localization, and a local property.

(c) Show that the "$= 0$" locus (as a property of a module element) of $m \in M$ is $D(\operatorname{ann}_R(m))$.

(a) The implication (i)⇒(ii) is clear from the definition of localization, and (ii)⇒(iii)⇒(iv) are tautologies. Suppose that $m \neq 0$. Then $\operatorname{ann}_R(m)$ is a proper ideal, so it is contained in some maximal ideal $\mathfrak{m}$. We claim that $m/1$ is nonzero in $M_{\mathfrak{m}}$. Indeed, $m/1$ is zero if and only if there is some $w \in R \smallsetminus \mathfrak{m}$ such that $wm = 0$, but by assumption this is impossible.

(b) The implication (i)⇒(ii) means preserved by localization, while (i)⇔(iii) means local property.

(c) Reviewing the argument from (a), we have $\frac{m}{1} = 0$ if and only if there is some $w \in W$ with $wm = 0$, which happens if and only if $R \smallsetminus \mathfrak{p} \cap \operatorname{ann}_R(m) = \varnothing$, which is equivalent to $\operatorname{ann}_R(m) \subseteq \mathfrak{p}$.

(2) Let R be a ring, M be a module.

(a) Show that the following are equivalent, and deduce that "$= 0$" (as a property of a module) is preserved by localization, and a local property.

(i) $M = 0$
(ii) $W^{-1}M = 0$ for all multiplicatively closed $W \subseteq R$;
(iii) $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$;
(iv) $M_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \in \operatorname{Max}(R)$.

[1] For example, two properties of a ring are "is reduced" or "is a domain".

[2] Hint: Go (i)⇒(ii)⇒(iii)⇒(iv)⇒(i). For the last, If $m \neq 0$, consider a maximal ideal containing $\operatorname{ann}_R(m)$.

(b) Prove[3] the Proposition.

> **(a)** Again (i)⇒(ii)⇒(iii)⇒(iv) are clear. If $M \neq 0$, take some nonzero $m \in M$. Then there is some $\mathfrak{m}$ such that $m/1$ is nonzero in $M_\mathfrak{m}$ so $M_\mathfrak{m} \neq 0$.
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> **(b)** Let $M = \sum_i Rm_i$. Since $M_\mathfrak{p} = \sum_i R_\mathfrak{p}\frac{m_i}{1}$, we have $M_\mathfrak{p} = 0$ if and only each $\frac{m_i}{1} = 0$, which happens if and only if $\mathfrak{p} \in \cap_i D(\mathrm{ann}_R(m_i))$. This equals $D(\cap_i D\mathrm{ann}_R(m_i)) = D(\mathrm{ann}_R(M))$. Then, we are considering the complement.

(3) More local properties

(a) Let R be a ring and $N \subseteq M$ modules. Show[4] that the following are equivalent, and deduce that $M = N$ for a submodule N is preserved by localization and a local property:

(i) $M = N$.
(ii) $W^{-1}M = W^{-1}N$ for all multiplicatively closed $W \subseteq R$;
(iii) $M_\mathfrak{p} = N_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$;
(iv) $M_\mathfrak{m} = N_\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(R)$.

(b) Let R be a ring. Show that the following are equivalent:

(i) R is reduced
(ii) $W^{-1}R$ is reduced for all multiplicatively closed $W \subseteq R$;
(iii) $R_\mathfrak{p}$ is reduced for all $\mathfrak{p} \in \mathrm{Spec}(R)$.
(iv) $R_\mathfrak{m}$ is reduced for all $\mathfrak{m} \in \mathrm{Max}(R)$.

> **(a)** Again (i)⇒(ii)⇒(iii)⇒(iv) are clear. If $N \subsetneqq M$, then $M/N \neq 0$, and by the above there is some $\mathfrak{m}$ such that $(M/N)_\mathfrak{m} \neq 0$. But $(M/N)_\mathfrak{m} \cong M_\mathfrak{m}/N_\mathfrak{m}$ so $N_\mathfrak{m} \subsetneqq M_\mathfrak{m}$.
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> **(b)** Suppose that R is reduced and let $W \subseteq R$ be multiplicatively closed. Take a nilpotent element r/w. Then $(r/w)^n = 0$ implies there is some $v \in W$ with $vr^n = 0$. Then $(vr)^n = 0$ so $vr = 0$ and $r/w = 0$ in $R_\mathfrak{p}$. Again (ii)⇒(iii)⇒(iv) are tautologies. Suppose that R is not reduced and take $r^n = 0$ with $r \neq 0$. By part (a), for every maximal ideal $\mathfrak{m}$ in $R_\mathfrak{m}$ we have $(r/1)^n = 0$, and for some maximal ideal we have $r/1 \neq 0$, so $R_\mathfrak{m}$ is not reduced.

(4) Not so local.

(a) Show that the property R is a domain is preserved by localization.

(b) Let K be a field and $R = K \times K$. Show that $R_\mathfrak{p}$ is a field for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Conclude that the property that R is a domain (or R is a field) is not a local property.

> **(a)** Suppose that R is a domain and $(a/u)(b/v) = 0$ in some $R_\mathfrak{p}$. Then there is some $w \notin \mathfrak{p}$ such that $wab = 0$, so $a = 0$ or $b = 0$, whence $a/u = 0$ or $b/v = 0$, so $R_\mathfrak{p}$ is a domain.
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> **(b)** The ring $K \times K$ has two prime ideals $0 \times K$ and $K \times 0$. The kernel of the localization map $(K \times K)_{0\times K}$ is the set of elements that are killed by some element not in $0 \times K$; i.e., the set of (a, b) such that there is some $(c, d) \in K^\times \times K$ with $(ac, bd) = (0, 0)$. This forces $a = 0$ and conversely, for an element $(0, b)$ we have $(0, b)(1, 0) = (0, 0)$, so this kernel is exactly $0 \times K$. Thus
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> $$(K \times K)_{0\times K} \cong \left(\frac{K \times K}{0 \times K}\right)_{0\times K} \cong K_0 \cong K.$$
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> Similarly for the other prime.

[3] Recall that if $M = \sum_i Rm_i$ is finitely generated then $W^{-1}M = \sum_i W^{-1}R\frac{m_i}{1}$ and that an element annihilates a module if and only if it annihilates every generator in a generating set.

[4] Hint: Consider M/N.

(5) More local properties, or not.
  (a) Let M be an R-module. Show that the property that M is finitely generated is preserved by localization but is not[5] a local property.
  (b) Let $R \subseteq S$ be an inclusion of rings. Show that the properties that $R \subseteq S$ is algebra-finite/integral/module-finite are preserved by localization on R: i.e., if one of these holds, the same holds for $W^{-1}R \subseteq W^{-1}S$ for any $W \subseteq R$ multiplicatively closed.
  (c) Let $R \subseteq S$ be an inclusion of rings, and $s \in S$. Show that the property that $s \in S$ is integral over R is a local property on R: i.e., this holds if and only if it holds for $\frac{s}{1} \in S_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$.
  (d) Is the property that $r \in R$ is a unit a local property?
  (e) Is the property that $r \in R$ is a zerodivisor a local property?
  (f) Is the property that $r \in R$ is nilpotent a local property?
  (g) Let $R \subseteq S$ be an inclusion of rings. Are the properties $R \subseteq S$ is algebra-finite/module-finite local properties on R?

(6) Let $\mathcal{P}$ be a local property of a ring, and $f_1, \dots, f_t \in R$ such that $(f_1, \dots, f_t) = R$. Show that if $\mathcal{P}$ holds for each R_{f_i}, then $\mathcal{P}$ holds for R.

[5]Hint: Consider $\bigoplus_{\alpha \in \mathbb{C}} \mathbb{C}[X]/(X - \alpha)$