DEFINITION: Let $\mathcal{P}$ be a property[1] of a ring. We say that

- $\mathcal{P}$ is **preserved by localization** if

$$\mathcal{P} \text{ holds for } R \Longrightarrow \text{for every multiplicatively closed set } W,\ \mathcal{P} \text{ holds for } W^{-1}R.$$

- $\mathcal{P}$ is a **local property** if

$$\mathcal{P} \text{ holds for } R \Longleftrightarrow \text{for every prime ideal } \mathfrak{p} \in \operatorname{Spec}(R),\ \mathcal{P} \text{ holds for } R_{\mathfrak{p}}.$$

One defines **preserved by localization** and **local property** for properties of modules in the same way, or for properties of a ring element (where one considers $\frac{r}{1} \in W^{-1}R$ or $R_{\mathfrak{p}}$ in the right-hand side) or module element.

DEFINITION: The **support** of a module M is

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}.$$

PROPOSITION: If M is a finitely generated module, then $\operatorname{Supp}(M) = V(\operatorname{ann}_R(M))$.

(1) Let R be a ring, M be a module, and $m \in M$.

(a) Show that[2] the following are equivalent:

(i) $m = 0$ in M;
(ii) $\frac{m}{1} = 0$ in $W^{-1}M$ for all multiplicatively closed $W \subseteq R$;
(iii) $\frac{m}{1} = 0$ in $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$;
(iv) $\frac{m}{1} = 0$ in $M_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Max}(R)$.

(b) Deduce that "$= 0$" (as a property of a module element) is preserved by localization, and a local property.

(c) Show that the "$= 0$" locus (as a property of a module element) of $m \in M$ is $D(\operatorname{ann}_R(m))$.

(2) Let R be a ring, M be a module.

(a) Show that the following are equivalent, and deduce that "$= 0$" (as a property of a module) is preserved by localization, and a local property.

(i) $M = 0$
(ii) $W^{-1}M = 0$ for all multiplicatively closed $W \subseteq R$;
(iii) $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$;
(iv) $M_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \in \operatorname{Max}(R)$.

(b) Prove[3] the Proposition.

(3) More local properties

(a) Let R be a ring and $N \subseteq M$ modules. Show[4] that the following are equivalent, and deduce that $M = N$ for a submodule N is preserved by localization and a local property:

(i) $M = N$.
(ii) $W^{-1}M = W^{-1}N$ for all multiplicatively closed $W \subseteq R$;
(iii) $M_{\mathfrak{p}} = N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$;
(iv) $M_{\mathfrak{m}} = N_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Max}(R)$.

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[1] For example, two properties of a ring are "is reduced" or "is a domain".

[2] Hint: Go (i)⇒(ii)⇒(iii)⇒(iv)⇒(i). For the last, If $m \neq 0$, consider a maximal ideal containing $\operatorname{ann}_R(m)$.

[3] Recall that if $M = \sum_i Rm_i$ is finitely generated then $W^{-1}M = \sum_i W^{-1}R\frac{m_i}{1}$ and that an element annihilates a module if and only if it annihilates every generator in a generating set.

[4] Hint: Consider M/N.

(b) Let R be a ring. Show that the following are equivalent:
   (i) R is reduced
   (ii) $W^{-1}R$ is reduced for all multiplicatively closed $W \subseteq R$;
   (iii) $R_\mathfrak{p}$ is reduced for all $\mathfrak{p} \in \operatorname{Spec}(R)$.
   (iv) $R_\mathfrak{m}$ is reduced for all $\mathfrak{m} \in \operatorname{Max}(R)$.

(4) Not so local.
   (a) Show that the property R is a domain is preserved by localization.
   (b) Let K be a field and $R = K \times K$. Show that $R_\mathfrak{p}$ is a field for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Conclude that the property that R is a domain (or R is a field) is not a local property.

(5) More local properties, or not.
   (a) Let M be an R-module. Show that the property that M is finitely generated is preserved by localization but is not[5] a local property.
   (b) Let $R \subseteq S$ be an inclusion of rings. Show that the properties that $R \subseteq S$ is algebra-finite/integral/module-finite are preserved by localization on R: i.e., if one of these holds, the same holds for $W^{-1}R \subseteq W^{-1}S$ for any $W \subseteq R$ multiplicatively closed.
   (c) Let $R \subseteq S$ be an inclusion of rings, and $s \in S$. Show that the property that $s \in S$ is integral over R is a local property on R: i.e., this holds if and only if it holds for $\frac{s}{1} \in S_\mathfrak{p}$ over $R_\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$.
   (d) Is the property that $r \in R$ is a unit a local property?
   (e) Is the property that $r \in R$ is a zerodivisor a local property?
   (f) Is the property that $r \in R$ is nilpotent a local property?
   (g) Let $R \subseteq S$ be an inclusion of rings. Are the properties $R \subseteq S$ is algebra-finite/module-finite local properties on R?

(6) Let $\mathcal{P}$ be a local property of a ring, and $f_1, \dots, f_t \in R$ such that $(f_1, \dots, f_t) = R$. Show that if $\mathcal{P}$ holds for each R_{f_i}, then $\mathcal{P}$ holds for R.

[5]Hint: Consider $\bigoplus_{\alpha \in \mathbb{C}} \mathbb{C}[X]/(X - \alpha)$